Sisterhood
in the Gale-Shapley Matching Algorithm

Yannai A. Gonczarowski

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The Hebrew University of Jerusalem

June 3, 2013

Joint work with Ehud Friedgut

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The Stable Matching Problem

- Two disjoint finite sets to be matched: women $W$ and men $M$.
  - Assume 1-to-1 for now.
  - Assume $|W| = |M|$ for now.
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  - Assume a strict order of preference for each woman over all men and vice versa.
  - Assume no blacklists for now.
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- Preferences for each woman and for each man.
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  - Assume no blacklists for now.

- The goal: a stable matching.
  - If $w$ and $m$ are matched, and if $w'$ and $m'$ are matched, then $w$ and $m'$ should not both prefer each other over their spouses.
Gale and Shapley (1962)

The following algorithm yields a stable matching.

1. On each night, every man serenades under the window of the woman he prefers most out of those who have not yet rejected him.
The Gale-Shapley Deferred-Acceptance Algorithm

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2. On each night, every woman rejects all the men serenading under her window, except for the one she prefers most among them.

3. When no more rejections occur, each woman is matched with the man serenading under her window.
Gender Duality and Manipulation Incentives

Gale and Shapley (1962)
No stable matching is better for any man.

McVitie and Wilson (1971)
No stable matching is worse for any woman.
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No subset of men can lie in a way that would make them all better off lying.

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Generally, there exists a woman who would be better off lying.
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Note: the latter two do not follow from the former two.
Example: Manipulation by Women

<table>
<thead>
<tr>
<th>Men’s Preferences</th>
<th>Women’s Preferences</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_1 )</td>
<td>( w_2 )  ( w_1 )  ( w_3 )</td>
</tr>
<tr>
<td>( m_2 )</td>
<td>( w_1 )  ( w_2 )  ( w_3 )</td>
</tr>
<tr>
<td>( m_3 )</td>
<td>( w_1 )  ( w_3 )  ( w_2 )</td>
</tr>
<tr>
<td>( w_1 )</td>
<td>( m_1 )  ( m_2 )  ( m_3 )</td>
</tr>
<tr>
<td>( w_2 )</td>
<td>( m_2 )  ( m_1 )</td>
</tr>
<tr>
<td>( w_3 )</td>
<td>any</td>
</tr>
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</tr>
<tr>
<td>( m_3 )</td>
<td>( w_1 ) ( w_3 ) ( w_2 )</td>
</tr>
<tr>
<td>( w_1 )</td>
<td>( m_1 ) ( m_2 ) ( m_3 )</td>
</tr>
<tr>
<td>( w_2 )</td>
<td>( m_2 ) ( m_1 )</td>
</tr>
<tr>
<td>( w_3 )</td>
<td>any</td>
</tr>
</tbody>
</table>

1. \( w_1 \) to \( m_2, m_3 \)
2. \( w_2 \) to \( m_1 \)
3. \( w_3 \) is unharmed.

\( w_1 \) made \( w_2 \) "give up" \( m_1 \) by making sure \( w_2 \) is approached by someone \( w_2 \) prefers better.
Example: Manipulation by Women

**Men’s Preferences**

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<tr>
<th>$m_1$</th>
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</tr>
</thead>
<tbody>
<tr>
<td>$m_2$</td>
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<td>$w_3$</td>
</tr>
<tr>
<td>$m_3$</td>
<td>$w_1$</td>
<td>$w_3$</td>
<td>$w_2$</td>
</tr>
</tbody>
</table>

**Women’s Preferences**

<table>
<thead>
<tr>
<th>$w_1$</th>
<th>$m_1$ &gt; $m_2$ &gt; $m_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_2$</td>
<td>$m_2$ &gt; $m_1$</td>
</tr>
<tr>
<td>$w_3$</td>
<td>any</td>
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- $w_1$ improved her match, but so did $w_2$; and $w_3$ is unharmed.
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<td>$w_1$</td>
</tr>
<tr>
<td>$m_3$</td>
<td>$w_1$</td>
</tr>
</tbody>
</table>

| $w_1$ | $m_1 > m_2 > m_3 > m_2$ |
| $w_2$ | $m_2 > m_1$ |
| $w_3$ | any |

\[ 1 \quad m_2, m_3 \]
\[ 2 \quad m_2 \]

\[ m_1 \]
\[ m_3 \]
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\[
\begin{align*}
&\begin{array}{c}
  w_1 \\
1 & m_2, m_3 \\
2 & m_2 \\
2 & m_3 \\
3 & m_1, m_3
\end{array} \\
&\begin{array}{c}
  w_2 \\
1 & m_1 \\
2 & m_1, m_2 \\
3 & m_2
\end{array} \\
&\begin{array}{c}
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\end{array}
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<td>$w_1$</td>
</tr>
<tr>
<td>$m_3$</td>
<td>$w_1$</td>
</tr>
<tr>
<td>$w_1$</td>
<td>$m_1 &gt; m_{23} &gt; m_{32}$</td>
</tr>
<tr>
<td>$w_2$</td>
<td>$m_2 &gt; m_1$</td>
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Sisterhood Theorem

Assume that a subset of the women declare false orders of preference for themselves.

We examines two runs of the Gale-Shapley algorithm:
- **OA** — according to everyone’s true preferences; yields the matching $O$.
- **NA** — according to the liars’ false preferences, and everyone else’s true preferences; yields the matching $N$. 
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**Theorem (Sisterhood)**

*Under the above conditions, if all *lying* women are weakly better off, then:*

1. All women are weakly better off.
2. All men are weakly worse off.*
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**Theorem (Sisterhood)**

*Under the above conditions, if all lying women are weakly better off, then:*

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No such “hoodness” exists within any other subset of \( W \cup M \). Indeed, when even a single man lies and is weakly b/o, some women and men may be b/o, and some others — w/o.
Observation

*If every lying woman \( w \) lies in an optimal way (i.e. the lies constitute a Nash Equilibrium in the lying game), then the new matching is stable.*
An Easy Proof?

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If every lying woman $w$ lies in an optimal way (i.e. the lies constitute a Nash Equilibrium in the lying game), then the new matching is stable.

Proof.

The new matching is obviously stable w.r.t. the new preferences. It is thus enough to consider couples in which at least one liar participates.

\[
\begin{array}{cc}
  w & w' \\
  m & m'
\end{array}
\]
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  \mid & \mid \\
  m & m'
\end{array}
\]

So, what’s the problem? Why would someone lie in a non-optimal way? Why do we care about non-equilibrium?
When a Lie Need Not be Optimal

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</tr>
<tr>
<td>( m_3 )</td>
<td>( w_3 )</td>
</tr>
<tr>
<td>( m_4 )</td>
<td>( w_1 )</td>
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</tr>
<tr>
<td>$m_4$</td>
<td>$w_1$</td>
</tr>
<tr>
<td>$w_1$</td>
<td>$m_3 &gt; m_1 &gt; m_2, m_4$</td>
</tr>
<tr>
<td>$w_2$</td>
<td>$m_3 &gt; m_1 &gt; m_2, m_4$</td>
</tr>
<tr>
<td>$w_3$</td>
<td>$m_2 &gt; m_1 &gt; m_3$</td>
</tr>
<tr>
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<td>any</td>
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2. $m_2$
3. $m_3$
4. $m_4, m_1$
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</tr>
</thead>
<tbody>
<tr>
<td>$m_2$</td>
<td>$w_2$</td>
<td>$w_3$</td>
<td>any</td>
<td>any</td>
</tr>
<tr>
<td>$m_3$</td>
<td>$w_3$</td>
<td>$w_2$</td>
<td>$w_1$</td>
<td>$w_4$</td>
</tr>
<tr>
<td>$m_4$</td>
<td>$w_1$</td>
<td>$w_4$</td>
<td>any</td>
<td>any</td>
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</table>

**Women’s Preferences**

| $w_1$ | $m_3 > m_1 > m_2, m_4$ |
| $w_2$ | $m_3 > m_1 > m_2, m_4$ |
| $w_3$ | $m_2 > m_1 > m_3$      |
| $w_4$ | any                     |
When a Lie Need Not be Optimal

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<table>
<thead>
<tr>
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<th>$w_1$</th>
<th>$w_3$</th>
<th>$w_2$</th>
<th>$w_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_2$</td>
<td>$w_2$</td>
<td>$w_3$</td>
<td>any</td>
<td>any</td>
</tr>
<tr>
<td>$m_3$</td>
<td>$w_3$</td>
<td>$w_2$</td>
<td>$w_1$</td>
<td>$w_4$</td>
</tr>
<tr>
<td>$m_4$</td>
<td>$w_1$</td>
<td>$w_4$</td>
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<td>any</td>
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</tbody>
</table>

**Women's Preferences**

<table>
<thead>
<tr>
<th>$w_1$</th>
<th>$m_3 &gt; m_4 &gt; m_2, m_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_2$</td>
<td>$m_3 &gt; m_1 &gt; m_2, m_4$</td>
</tr>
<tr>
<td>$w_3$</td>
<td>$m_2 &gt; m_1 &gt; m_3$</td>
</tr>
<tr>
<td>$w_4$</td>
<td>any</td>
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</table>

1. $m_4, m_1$
2. $m_1$

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<tr>
<th>m1</th>
<th>w₁</th>
<th>w₃</th>
<th>w₂</th>
<th>w₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>m₂</td>
<td>w₂</td>
<td>w₃</td>
<td>any</td>
<td>any</td>
</tr>
<tr>
<td>m₃</td>
<td>w₃</td>
<td>w₂</td>
<td>w₁</td>
<td>w₄</td>
</tr>
<tr>
<td>m₄</td>
<td>w₁</td>
<td>w₄</td>
<td>any</td>
<td>any</td>
</tr>
</tbody>
</table>

Women’s Preferences

<table>
<thead>
<tr>
<th>w₁</th>
<th>m₃  &gt; m₄ &gt; m₂, m₁</th>
</tr>
</thead>
<tbody>
<tr>
<td>w₂</td>
<td>m₃  &gt; m₃ &gt; w₂, m₄</td>
</tr>
<tr>
<td>w₃</td>
<td>m₂  &gt; m₁ &gt; m₃</td>
</tr>
<tr>
<td>w₄</td>
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<th>m1</th>
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<th>w3</th>
<th>w2</th>
<th>w4</th>
</tr>
</thead>
<tbody>
<tr>
<td>m2</td>
<td>w2</td>
<td>w3</td>
<td>any</td>
<td>any</td>
</tr>
<tr>
<td>m3</td>
<td>w3</td>
<td>w2</td>
<td>w1</td>
<td>w4</td>
</tr>
<tr>
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<td>any</td>
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Women’s Preferences

<table>
<thead>
<tr>
<th>w1</th>
<th>m3 &gt; m4 &gt; m2, m1</th>
</tr>
</thead>
<tbody>
<tr>
<td>w2</td>
<td>m3 &gt; m1 &gt; m2, m4</td>
</tr>
<tr>
<td>w3</td>
<td>m2 &gt; m1 &gt; m3</td>
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<td>w4</td>
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1. $m_4, m_1$
2. $m_1$
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Women’s Preferences

<table>
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When a Lie Need Not be Optimal

<table>
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<th>Men’s Preferences</th>
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<tr>
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\( m_4 \), \( m_1 \)
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\( m_2 \)

improved her match, but so did \( w_2 \); and \( w_3 \) is unharmed.
When a Lie Need Not be Optimal

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Sisterhood in the Gale-Shapley Matching Algorithm

Yannai A. Gonczarowski (HUJI)
When a Lie Need Not be Optimal

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Yannai A. Gonczarowski (HUJI)
Sisterhood in the Gale-Shapley Matching Algorithm
June 3, 2013 8 / 18
When a Lie Need Not be Optimal (cont.)

In this example:

- The truth is an optimal strategy for any coalition not including \( w_1 \).
In this example:

- The truth is an optimal strategy for any coalition not including $w_1$.
- No strategy for $w_1$ is better than the truth if all other women respond optimally to it.
When a Lie Need Not be Optimal (cont.)

In this example:

- The truth is an optimal strategy for any coalition not including \( w_1 \).
- No strategy for \( w_1 \) is better than the truth if all other women respond optimally to it.
- Thus, in no Nash equilibrium is any woman better-matched than according to all the true preferences.
When a Lie Need Not be Optimal (cont.)

In this example:

- The truth is an optimal strategy for any coalition not including $w_1$.

- No strategy for $w_1$ is better than the truth if all other women respond optimally to it.

- Thus, in no Nash equilibrium is any woman better-matched than according to all the true preferences.

- There exists a strategy for $w_1$ and $w_2$ that is better for both than the truth, but which is out-of-equilibrium due to $w_2$ lying suboptimally.
An Easy Proof? Take II

Roth (private communication, Dec. 2007)

1. If women can do better than to state their true preferences, they can do so by truncating their preferences.
2. Truncating preferences is the opposite of extending preferences.
3. When any woman extends her preferences, it harms the other women.
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But this still proves only the optimal lie case... (Indeed, $w_2$'s lie from the previous example is not equivalent to any truncation of her preferences.)
A Make-Believe Proof

Definition (Rejecter)

A woman $w \in W$ is said to be a rejecter if she rejects $N(w)$ on some night during OA. Denote that night by $T(w)$, and the man who serenades under $w$’s window on that night, but whom she does not reject then — $B(w)$. 
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1. $w \in W$ is worse off $\Rightarrow O(w)$ is better off.
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1. \( w \in W \) is worse off \( \Rightarrow \) \( O(w) \) is better off.
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5. $w \in W$ is a rejecter $\Rightarrow$
   i. $N(B(w))$ is a rejecter.
   ii. $T(w) > T(N(B(w)))$. 

Yannai A. Gonczarowski (HUJI)  
Sisterhood in the Gale-Shapley Matching Algorithm  
June 3, 2013 11 / 18
Connecting the Dots

1. \( w \in W \) is worse off \( \Rightarrow O(w) \) is better off.
2. \( m \in M \) is better off \( \Rightarrow N(m) \) is a rejecter.
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Assume some part does not hold
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$$w_2 = N(B(w_1))$$
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Assume some part does not hold \( \Rightarrow \) there exists a rejecter \( w_1 \).

\[
\begin{align*}
  w_2 &= N(B(w_1)) \\
  w_3 &= N(B(w_2))
\end{align*}
\]
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  w_2 & = N(B(w_1)) \\
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  & \vdots \\
  w_{i-2} & = N(B(w_{i-3})) \\
  w_{i-1} & = N(B(w_{i-2})) \\
  w_i & = N(B(w_{i-1}))
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\]
One-To-Many and Many-To-Many Matchings

- What's better off?
- What's worse off?
- We still assume total preferences over individuals.
- For a person $p$, denote $O(p) = (o_{p1}, \ldots, o_{pn_p})$ and $N(p) = (n_{p1}, \ldots, n_{pn_p})$. Lower index = higher on $p$'s list.

**Definition (Improvement)**
A person $p$ is said to be weakly better off if for each $1 \leq i \leq n_p$, $p$ weakly prefers $n_{pi}$ over $o_{pi}$.

**Definition (Worsening)**
A person $p$ is said to have gained only worse matches if $p$ prefers every member of $O(p)$ over every member of $N(p) \setminus O(p)$.

By Gusfield and Irving (1989): These are dual total orders over equivalence classes of stable matchings.
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Sisterhood Theorem — Polygamous Case

Theorem (Sisterhood)

If all lying women are weakly better off, then:

1. All women are weakly better off.
2. All men have gained only worse matches.
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- A reduction proof works only when men are monogamous.
Sisterhood Theorem — Polygamous Case

Theorem (Sisterhood)

If all lying women are weakly better off, then:
1. All women are weakly better off.
2. All men have gained only worse matches.

- A reduction proof works only when men are monogamous.
- Otherwise, we must revisit the proof.
Adapting the Proof

Definition (Rejecter )

A woman $w$ is said to be a rejecter if she rejects $N(w)$ during $OA$.

1. $w \in W$ is worse off $\Rightarrow O(w)$ is better off.
2. $m \in M$ is better off $\Rightarrow N(m)$ is a rejecter.
3. $w \in W$ is a rejecter $\Rightarrow w$ is worse off.
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   i. $B(w)$ prefers $N(B(w))$ over $w$.
   
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Adapting the Proof

Definition (Rejecter / Rejectee)

A woman \(w\) is said to be a rejecter if she rejects any of \(N(w)\) during OA. Denote the set of all such rejected members of \(N(w)\) by \(R(w)\). A man \(m\) is said to be a rejectee if there exists a rejecter \(w \in N(m)\) such that \(m \in R(w)\).

1. \(w \in W\) is worse off \(\Rightarrow\) \(O(w)\) is better off.
2. \(m \in M\) is better off \(\Rightarrow\) \(N(m)\) is a rejecter.
3. \(w \in W\) is a rejecter \(\Rightarrow\) \(w\) is worse off.
4. \(w \in W\) is a rejecter \(\Rightarrow\)
   - \(B(w)\) prefers \(N(B(w))\) over \(w\).
5. \(w \in W\) is a rejecter \(\Rightarrow\)
   - \(N(B(w))\) is a rejecter.
   - \(T(w) > T(N(B(w)))\).
Adapting the Proof

Definition (Rejecter / Rejectee)

A woman \( w \) is said to be a rejecter if she rejects any of \( N(w) \) during \( OA \). Denote the set of all such rejected members of \( N(w) \) by \( R(w) \). A man \( m \) is said to be a rejectee if there exists a rejecter \( w \in N(m) \) such that \( m \in R(w) \).

1. \( w \in W \) is worse off \( \implies \) \( O(w) \) is better off.
2. \( m \in M \) is better off \( \implies \) \( N(m) \) is a rejecter.
3. \( w \in W \) is a rejecter \( \implies \) \( w \) is worse off.
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   i. \( B(w) \) prefers \( N(B(w)) \) over \( w \).
   
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5. \( w \in W \) is a rejecter \( \implies \)
   
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1. $w \in W$ isn’t weakly b/o $\Rightarrow$ $O(w)$ is better off.
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Sisterhood in the Gale-Shapley Matching Algorithm  
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15 / 18
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3. \( w \in W \) is a rejecter \( \Rightarrow \) \( w \) is worse off.
4. \( w \in W \) is a rejecter \( \Rightarrow \)
   - \( N(B(w)) \) is a rejecter.
   - \( \exists w' \in W \) s.t. \( T(w') > T(N(B(w))) \).
5. \( w \in W \) is a rejecter \( \Rightarrow \)
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**Definition (Rejecter / Rejectee)**

A woman \( w \) is said to be a *rejecter* if she rejects *any* of \( N(w) \) during \( OA \). Denote the set of all such rejected members of \( N(w) \) by \( R(w) \). A man \( m \) is said to be a *rejectee* if there exists a rejecter \( w \in N(m) \) such that \( m \in R(w) \).

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4. \( w \in W \) is a rejecter \( \Rightarrow \\
   \qquad \text{i} \quad B(w) \) prefers \( N(B(w)) \) over \( w \).
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   i. \( B(w, r) \) serenades under \( w \)’s window during OA on the night on which she rejects \( r \), but is not rejected then.
   ii. \( B(w, r) \) prefers all of \( N(B(w, r)) \) over \( w \).
5. \( w \in W \) is a rejecter \( \Rightarrow \)
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A woman \( w \) is said to be a **rejecter** if she rejects any of \( N(w) \) during \( OA \). Denote the set of all such rejected members of \( N(w) \) by \( R(w) \). A man \( m \) is said to be a **rejectee** if there exists a rejecter \( w \in N(m) \) such that \( m \in R(w) \).

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Blacklists and Mismatched Quotas

The Sisterhood theorem still holds under the following definitions: (Proof by reduction to the previous case.)

Definition (Improvement)
A person $p$ is said to be weakly better off if:
1. $|N(p)|$ contains no-one who is blacklisted by $p$.
2. $|N(p)| \geq |O(p)|$.
3. For each $1 \leq i \leq |O(p)|$, $p$ weakly prefers $n_p_i$ over $o_p_i$.

Definition of Worsening is Unchanged
A person $p$ is said to have gained only worse matches if $p$ prefers every member of $O(p)$ over every member of $N(p) \setminus O(p)$.

(∗Does not require $|N(p)| \leq |O(p)|$, but equality will follow.)
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**Definition (Improvement)**

A person $p$ is said to be *weakly better off* if:

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A Few Sample Corollaries

Under the above conditions,

$$1 | N(p) | = | O(p) |$$

for each person $$p \in W \cup M$$.

For an innocent person $$p$$, if $$| N(p) | < n_p$$, then $$N(p) = O(p)$$.

Corollary

If $$| L | = 1$$, and the lying woman is (strictly) better off, then so is some innocent woman.

Corollary

If all women have the same order of preference, then under the above conditions the matching must remain unchanged. Therefore, in this case there is no "significant" incentive for any subset of women to lie, even for the sake of one of them.
A Few Sample Corollaries

A Rural-Hospitals-type Theorem

Under the above conditions,

1. \(|N(p)| = |O(p)|\) for each person \(p \in W \cup M\).

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Questions?

Thank you!