Manipulation of Stable Matchings using Minimal Blacklists

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The Stable Matching Problem (Gale&Shapley 1962)

- Two disjoint finite sets: women $W$ and men $M$.
- One-to-one.
- Assume $|W| = |M|$ for now.
- A preferences list for each woman and for each man.
- Strictly ordered.
- The blacklist is the set of those not on the preference list.
- The goal: a stable matching.
- $M$-rational: No man is matched with a woman from his blacklist.
- $W$-rational: No woman is matched with a man from her blacklist.
- If $w$ and $m$ are not matched, then at least one of them prefers their spouse (or lack thereof) over the other.

Roth (2002) "Successful matching mechanisms produce stable outcomes."
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A stable matching exists for every profile of preference lists.
An efficient algorithm for finding the (unique) $M$-optimal one.

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The $M$-optimal stable matching = the $W$-worst stable matching.

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Full-Side Manipulation

The coalition of all men can force any \( W \)-rational perfect matching as the \( M \)-optimal stable one. (Distinct top choices.) Gale and Sotomayor (1985)

The coalition of all women can force the \( W \)-optimal stable matching as the \( M \)-optimal one by truncating preference lists.

- Requires blacklists.
- Possibly long blacklists.
- Possibly each of size \(|M|-1\).
- Conspiracy is painfully obvious.

Gusfield and Irving (1989)

No results are known regarding achieving this by any means other than such preference-list truncation, i.e. by also permuting preference lists.
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Define $n \triangleq |W| = |M|$. The women may force the $W$-optimal stable matching as the $M$-optimal one, using a profile of preference lists with average blacklist size no more than . . .
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1. $c$
2. $O(\log n)$
3. $O(n^{1/c})$
4. $O\left(\frac{n}{\log n}\right)$
5. $\frac{n}{c}$
6. $n - c$

By truncation
A Short Poll

Define $n \equiv |W| = |M|$. The women may force the $W$-optimal stable matching as the $M$-optimal one, using a profile of preferences with average blacklist size no more than . . .

\[
\begin{align*}
1 & \quad c \\
2 & \quad O(\log n) \\
3 & \quad O(n^{1/c}) \\
4 & \quad O\left(\frac{n}{\log n}\right) \\
5 & \quad \frac{n}{c} \\
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Answering Gusfield and Irving’s Open Question

Summary of Main Result (Weak Version)

- The women may force any $M$-rational perfect matching as the unique stable matching, using a profile of preference lists in which at most half of the women have blacklists, and in which the average blacklist size is less than 1.
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- Each of these bounds is tight: it cannot be improved upon.

- This profile of preference lists may be computed efficiently.

- Generally, many such profiles of preference lists exist.

A far more “inconspicuous” manipulation, esp. if preference-list lengths are bounded (e.g. New York High School Match). If women pay a price for every man they blacklist, then order-of-magnitude improvement.
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Unbalanced Markets and Partial Matchings

• When there are less women than men (and all women are to be matched), no blacklists are required whatsoever.

• When there are more women than men (or if not all women are to be matched), each to-be-unmatched woman may have to blacklist as many as all men.

• Ashlagi et al. (2013) show a similar phase change w.r.t. the expected ranking of the stable partners of each participant on this participant's preference list in a random market. ($\log n$ vs. $n / \log n$)

• (cf. the shoe market.)

• Completely different proofs.
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Improved Insight into Matching Markets

Both phase-change results lead to a similar conclusion in different senses:

The preferences of the smaller side of the market (even if only slightly smaller) play a far more significant role than may be expected in determining the stable matchings, and those of the larger side — a considerably insignificant one.
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In a sense, our results extend this qualitative statement from a random matching market to any matching market.

More generally: our results shed light on the question of how much, if at all, do given preferences for one side \textit{a priori} impose limitations on the set of stable matchings under various conditions.
“Example Insight”: Goods Allocation Problems

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- A concrete supporting argument from our result: if goods have no preferences, then many lotteries = all *buyer-rational* matchings are possible*; single lottery = random serial (buyer) dictatorship $\Rightarrow$ Pareto-efficient outcome.
Theorem (Manipulation with Minimal Blacklists)

Define \( n \equiv |W| = |M| \). Let \( P_M \) be a profile of preference lists. For every \( M \)-rational perfect matching \( \mu \), there exists a profile \( P_W \) of preference lists for \( W \), s.t. all the following hold.

1. The unique stable matching, given \( P_W \) and \( P_M \), is \( \mu \).
2. The blacklists in \( P_W \) are pairwise disjoint, i.e. no man appears in more than one blacklist.
3. \( n_b \), the number of women who have nonempty blacklists in \( P_W \), is at most \( n \).
4. The combined size of all blacklists in \( P_W \) is at most \( n - n_b \), i.e. at most the number of women who have empty blacklists.

Furthermore, \( P_W \) can be computed in worst-case \( O(n^3) \) time, best-case \( O(n^2) \) time and average-case (assuming \( \mu \) is uniformly distributed given \( P_M \)) \( O(n^2 \log n) \) time.
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Full Result for Balanced Markets

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4. The combined size of all blacklists in $\mathcal{P}_W$ is at most $n - n_b$.

Examples of blacklist sizes for $n = 8$:

7 0 0 0 0 0 0 0 ($n_b = 1$)
Tradeoff: #Blacklists vs. Combined Blacklist Size

3. \( n_b \), the number of women who have nonempty blacklists in \( \mathcal{P}_W \), is at most \( \frac{n}{2} \).

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Examples of blacklist sizes for \( n = 8 \):

\[
\begin{align*}
7 & 0 0 0 0 0 0 0 \quad (n_b = 1) \\
1 & 1 1 1 0 0 0 0 \quad (n_b = 4)
\end{align*}
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- $7\ 0\ 0\ 0\ 0\ 0\ 0\ 0$  ($n_b = 1$)
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Examples of blacklist sizes for \( n = 8 \):

- 7 0 0 0 0 0 0 0 \( (n_b = 1) \)
- 1 1 1 1 0 0 0 0 \( (n_b = 4) \)
- 4 2 0 0 0 0 0 0 \( (n_b = 2) \)
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Examples of blacklist sizes for $n = 8$:

7 0 0 0 0 0 0 0 \quad (n_b = 1)
1 1 1 1 0 0 0 0 \quad (n_b = 4)
4 2 0 0 0 0 0 0 \quad (n_b = 2)
4 1 0 0 0 0 0 0 \quad (n_b = 2)
3 1 1 0 0 0 0 0 \quad (n_b = 3)
\vdots
Tradeoff: #Blacklists vs. Combined Blacklist Size

3. \( n_b \), the number of women who have nonempty blacklists in \( \mathcal{P}_W \), is at most \( \frac{n}{2} \).

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Examples of blacklist sizes for \( n = 8 \):

- 7 0 0 0 0 0 0 0 \( (n_b = 1) \)
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- 4 2 0 0 0 0 0 0 \( (n_b = 2) \)
- 4 1 0 0 0 0 0 0 \( (n_b = 2) \)
- 3 1 1 0 0 0 0 0 \( (n_b = 3) \)
- ...

Tightness

Each of these is the optimal solution for some \( \mathcal{P}_M \) and \( \mu \).
Tradeoff: \#Blacklists vs. Combined Blacklist Size

\begin{enumerate}
\item $n_b$, the number of women who have nonempty blacklists in $\mathcal{P}_W$, is at most $\frac{n}{2}$.
\item The combined size of all blacklists in $\mathcal{P}_W$ is at most $n - n_b$.
\end{enumerate}

Examples of blacklist sizes for $n = 8$:

- $7 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0$ ($n_b = 1$)
- $1 \ 1 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0$ ($n_b = 4$)
- $4 \ 2 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0$ ($n_b = 2$)
- $4 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0$ ($n_b = 2$)
- $3 \ 1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0$ ($n_b = 3$)
- ...

Tightness

Each of these is the optimal solution for some $\mathcal{P}_M$ and $\mu$. 
The Gale-Shapley Deferred-Acceptance Algorithm

A version modelled after Dubins and Freedman’s (1981)

The following algorithm yields the $M$-optimal stable matching.
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The (unique) $M$-optimal matching is always reached, regardless of the arbitrary choices made during the run.
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3. On each *night*, choose an arbitrary man scheduled for rejection. He moves to serenade under the window of the woman next on his preference list, if such woman exists.

4. When no men are scheduled for rejection, the algorithm terminates. Each woman is matched with the man serenading under her window; everyone else is unmatched.

The (unique) $M$-optimal matching is always reached, regardless of the arbitrary choices made during the run.
## Tightness Overview

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2. $m_1$
3. $m_2$
4. $m_3$
Tightness Overview

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Yannai A. Gonczarowski (HUJI&MSR)  Manipulation of Stable Matchings using Minimal Blacklists  July 29, 2014
Tightness Overview

\[
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& w_2 & w_3 & w_4 & w_1 \\
\hline
m_1 & > & > & > & \\
m_2 & w_3 & w_4 & w_1 & w_2 \\
m_3 & w_4 & w_1 & w_2 & w_3 \\
m_4 & w_1 & w_2 & w_3 & > \\
\end{array}
\]

\[
\begin{array}{l|llll}
& m_1 & > & > & > \\
\hline
w_1 & m_1 & > & > & > \\
w_2 & m_2 & m_1 & m_4 & m_3 \\
w_3 & m_3 & m_2 & m_1 & m_4 \\
w_4 & m_4 & m_3 & m_2 & m_1 \\
\end{array}
\]

(Blacklist: \( m_4, m_3, m_2 \))
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![Diagram with nodes and weights]
# Tightness Overview

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![Diagram](image-url)
### Tightness Overview

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Diagram:

1. $w_1$
2. $m_4$
3. $w_2$
4. $m_1\, m_4$
5. $m_1$
6. $w_3$
7. $m_2\, m_4$
8. $m_1$
9. $w_4$
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11. $m_3\, m_4$
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Tightness Overview

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# Tightness Overview

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- **w₁**: m₁  (Blacklist: m₄, m₃, m₂)
- **w₂**: m₂ > m₁ > m₄ > m₃
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- **w₄**: m₄ > m₃ > m₂ > m₁
Tightness Overview

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w1 | m1 (Blacklist: m4, m3, m2) |
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Tightness Overview

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Yannai A. Gonczarowski (HUJI&MSR) Manipulation of Stable Matchings using Minimal Blacklists July 29, 2014 13 / 18
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![Diagram](attachment:image.png)
Tightness Overview

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**Tightness Overview**

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Tightness Overview

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Yannai A. Gonczarowski (HUJI&MSR)  
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A Poll

Results

Overview

A Peek Into the Depths

Summary

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Construction Overview for an Easier Special Case

- Assume that the top choices of men are distinct, i.e. each man serenades under a unique window on the first night.
Construction Overview for an Easier Special Case

- Assume that the top choices of men are distinct, i.e. each man serenades under a unique window on the first night.
- We build a profile of preference lists for the women s.t. each woman prefers $\mu(w)$ most. $\Rightarrow \mu$ is $W$-optimal.
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- Assume that the top choices of men are distinct, i.e. each man serenades under a unique window on the first night.
- We build a profile of preference lists for the women s.t. each woman prefers $\mu(w)$ most. $\Rightarrow \mu$ is $W$-optimal.
- Choose a woman $\tilde{w}$ not serenaded-to by $\mu(\tilde{w})$, and have her blacklist her suitor $m$. 

- Let $m$ be repeatedly rejected until serenading to $\mu(m)$, who then rejects her suitor $m'$.
- Let $m'$ be repeatedly rejected until serenading to $\mu(m')$, who then rejects her suitor $\ldots$.
- Let $\mu(\tilde{w})$ be repeatedly rejected until serenading to $\tilde{w}$.
- Only $\tilde{w}$ blacklists anyone. More men have reached their intended partner than have been blacklisted.
- Naïve next step: choose some $\tilde{w}'$ and trigger another rejection cycle.
- Problem: all candidates for the role of $\tilde{w}'$ may have already rejected many men, whom we'd have to blacklist.
Construction Overview for an Easier Special Case

- Assume that the top choices of men are distinct, i.e. each man serenades under a unique window on the first night.
- We build a profile of preference lists for the women s.t. each woman prefers $\mu(w)$ most. $\Rightarrow \mu$ is $\mathcal{W}$-optimal.
- Choose a woman $\tilde{w}$ not serenaded-to by $\mu(\tilde{w})$, and have her blacklist her suitor $m$.
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- Let \( m' \) be repeatedly rejected until serenading to \( \mu(m') \), who then rejects her suitor \ldots
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Construction Overview for an Easier Special Case

- Assume that the top choices of men are distinct, i.e. each man serenades under a unique window on the first night.
- We build a profile of preference lists for the women s.t. each woman prefers $\mu(w)$ most. $\Rightarrow \mu$ is $W$-optimal.
- Choose a woman $\tilde{w}$ not serenaded-to by $\mu(\tilde{w})$, and have her blacklist her suitor $m$.
- Let $m$ be repeatedly rejected until serenading to $\mu(m)$, who then rejects her suitor $m'$.
- Let $m'$ be repeatedly rejected until serenading to $\mu(m')$, who then rejects her suitor . . .
- Let $\mu(\tilde{w})$ be repeatedly rejected until serenading to $\tilde{w}$.
- Only $\tilde{w}$ blacklists anyone. More men have reached their intended partner than have been blacklisted.
- Naïve next step: choose some $\tilde{w}'$ and trigger another rejection cycle.
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- **Solution:** show that it is possible to carefully “merge” the cycles, i.e. alter the preferences, “without blacklisting excessively-many men”, s.t. the “chain reaction” triggered by $\tilde{w}$ causes all rejections from both rejection cycles.
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- General case harder to analyse and slower to compute (and not online). “Conclusion”: the men inadvertently help the women in a sense by trying to force some matching.
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  General idea: follow the naïve construction; use these men as “placeholders” to initiate cycles without blacklisting.
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Questions?

Thank you!