Bulow-Klemperer-Style Results for Welfare Maximization in Two-Sided Markets

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Two Sided Markets ("Double Auctions")

- Each of $m_S$ sellers holds one item. All items identical.
- Each of $m_B$ (potential) buyers is interested in (any) one item.
- Each seller $j$ has private cost $s_j \geq 0$ for parting with her item.
- Each buyer $i$ has private value $b_i \geq 0$ for obtaining an item.
- A trade is a specification of a set of sellers (to part with their items) and an equal-sized set of buyers (to obtain these items). Efficient if maximizes the gains from trade:
  $$\sum_{\text{trading buyer } i} b_i - \sum_{\text{trading seller } j} s_j$$
- Goal: a mechanism (function from all values and costs to a trade + payment/charge for each participant) that is:
  - Individually rational (IR) — allows voluntary participation.
  - Incentive compatible (IC) — incentivizes truthful reporting.
  - Weakly budget balanced (BB) — does not lose money ("IR for the auctioneer").
  - Efficient — output trade efficient w.r.t. input costs/values.
Myerson and Satterthwaite’s Seminal Impossibility

**VCG** is a (generally applicable) IR, IC, efficient mechanism.
- Output efficient trade.
- Charge each trading buyer her minimum trading bid.
- Pay each trading seller her maximum trading bid.

**Example**

For one buyer with value $b = 10$ and one seller with cost $s = 9$:
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- Seller’s maximum trading bid is 10 $\Rightarrow$ seller paid 10.
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- Efficient trade is to trade the item. (Gains from trade $= 1$)
- Buyer’s minimum trading bid is $9 \Rightarrow$ buyer pays $9$.
- Seller’s maximum trading bid is $10 \Rightarrow$ seller paid $10$.
- **VCG with these inputs runs a deficit of $1$! $\Rightarrow$ VCG not BB.**
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For one buyer with value \( b = 10 \) and one seller with cost \( s = 9 \):
- Efficient trade is to trade the item. (Gains from trade = 1)
- Buyer’s minimum trading bid is 9 ⇒ buyer pays 9.
- Seller’s maximum trading bid is 10 ⇒ seller paid 10.
- **VCG with these inputs runs a deficit of 1! ⇒ VCG not BB.**

**Theorem (Myerson and Satterthwaite, 1983)**

Even for one seller and one buyer \((m_S = m_B = 1)\), there is no mechanism that is IR, IC, BB, and efficient.
The Road to a Positive Result

- “First best” efficiency infeasible!
- “Go to” mechanism design approach: maintain **feasibility** constraints (IR, IC, BB), relax efficiency.
  - Assume values and costs are independently drawn from some distribution, find feasible mechanism with optimal expected efficiency (“second best”).
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⇒ As in many mechanism-design settings, tradeoff between efficiency on the one hand, and on the other hand both mechanism simplicity and amount of knowledge required by mechanism.
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Will draw inspiration from the one-sided markets literature:
- A canonical setting: one seller with one item; $m$ buyers, each with a private value. Goal: maximize seller’s revenue.
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- Bulow-Klemperer (1996): if we can recruit one more similar buyer (=i.i.d. same distribution), we can “beat“ the tradeoff from the last slide: \( \exists \) a simple, prior-independent, feasible (IR & IC) mechanism that in the augmented market gives expected revenue \( \geq \) optimal revenue in the original market.
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Today: Bulow-Klemperer-style results for two-sided markets.
- “Beat the tradeoff”!® A simple, prior-independent, feasible (IR, IC, BB) mechanism that in an augmented market gives expected efficiency \( \geq \) optimal efficiency in the original market.
First Main Result

Setting:
- Market with $m_S$ sellers, $m_B$ buyers.
- Values and costs drawn i.i.d. from a distribution $F$.
- **Augmented market**: has one more buyer with value drawn independently from $F$. ($m_S$ sellers, $m_B+1$ buyers.)

Theorem (First Main Result — Informal)

There exists a simple, prior-independent (=does not require any information about $F$), IR, IC, BB mechanism such that this mechanism in the augmented market has expected gains from trade at least as high as the optimal-yet-infeasible VCG mechanism in the original market.
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- Same result also if adding a seller rather than a buyer.
  - Aesthetic preference to add buyer: same pre-trade welfare.
  - Same will hold also for all other results we’ll see today.
A Simple Mechanism

A special case: one seller, $m_B$ buyers.

- “Would have wanted” a second-price auction with the seller’s cost as the reserve price. (Efficient and BB!)
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Instead: “second-price auction with seller veto.”

- First, run among the buyers a second-price auction with no reserve. (Highest-value buyer wins, pays 2nd-highest value.)
- Next, the seller either accepts the result of the second-price auction (gets paid the 2nd-highest buyer value for her item) or vetoes it if the price is lower than her cost (resulting in no trade).

IC for the seller, but not efficient.

- No trade if highest value > seller’s cost > 2nd-highest value.

Nonetheless, we show that efficiency in the augmented market \( \geq \) optimal efficiency in the original market.
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A Simple Mechanism: Buyer Trade Reduction (BTR)

Inspired by McAfee’s (1992) classic

**Trade Reduction** mechanism:

- Sort buyer values in decreasing order, seller costs in increasing order.
- Calculate the efficient trade size $q$.
- Attempt to trade at the value of buyer $q + 1$ as the price.
- If a trading seller has higher cost than this price: reduce the trade between seller $q$ and buyer $q$.
- Trade the top $q - 1$ pairs, with all buyers paying the value of the reduced buyer and all sellers being paid the cost of the reduced seller.

Robust (prior-independent, IR, IC, BB) and anonymous.

**Theorem (First Main Result — Formal Restatement)**

$$\forall m_S, m_B, \forall F: \text{BTR}(m_S, m_B + 1) \geq \text{OPT}(m_S, m_B).$$
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\( p = 50 \quad 75 \)
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\hline
90 & 10 \\
70 & 20 \\
50 & 75 \\
40 & 95 \\
\end{array}
\]

\( \overline{PB} = 60 \) \( \overline{PS} = 45 \)
A Simple Mechanism: Buyer Trade Reduction (BTR)

Inspired by McAfee’s (1992) classic 

Trade Reduction mechanism:

- Sort buyer values in decreasing order, seller costs in increasing order.
- Calculate the efficient trade size $q$.
- Attempt to trade at the value of buyer $q+1$ as the price.
- If a trading seller has higher cost than this price: reduce the trade between seller $q$ and buyer $q$. Trade the top $q-1$ pairs, with all buyers paying the value of the reduced buyer and all sellers being paid the cost of the reduced seller.
- Robust (prior-independent, IR, IC, BB) and anonymous.

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$q = 3 \quad p_B = 60 \leftrightarrow 45 = p_s \quad q - 1 \quad p_B = 60$
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\[ p_B = 60 \leftarrow 45 = p_S \]

\[ q = 3 \]

\[ q - 1 \]

\[ \forall m_S, m_B, \forall F : \quad \text{BTR}(m_S, m_B + 1) \geq \text{OPT}(m_S, m_B). \]
Beyond I.I.D.

Proposition

Let $M$ be any anonymous robust deterministic mechanism. For all $\epsilon > 0$, for all $m_S$, $m_B$, for all $\ell, k$, there exist two distributions $F_S$ and $F_B$ such that $M(m_S + \ell, m_B + k) < \epsilon \cdot \text{OPT}(m_S, m_B)$.

Intuition: if any buyer value is rarely above any seller value (but lots of gains from trade when any is), then slim probability of another buyer value also above that seller value (needed for BTR to not reduce the trade). So, we henceforth assume that $F_B$ first-order stochastically dominates ($\text{FSD}$) $F_S$ to avoid such cases.
Beyond I.I.D.

A strong negative result for augmenting any market size with any number of sellers and/or buyers:

**Proposition**

Let $M$ be any anonymous robust deterministic mechanism. \(\forall \varepsilon > 0, \forall m_S, m_B, \forall \ell, k, \exists \) two distributions $F_S$ and $F_B$ s.t.

\[
M(m_S + \ell, m_B + k) < \varepsilon \cdot \text{OPT}(m_S, m_B).
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One Seller Stochastically Dominated by One Buyer
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Quantitative “miss”:

Theorem (Lower Bound for One Seller and One Buyer)

Let $M$ be any anonymous robust deterministic mechanism.  
$\exists$ two distributions $F_S$ and $F_B$, s.t. $F_B \ FSD \ F_S$ and for which

\[ M(1, 2) < \text{OPT}(1, 1). \]
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Qualitative “hit”:

Theorem (Upper Bound for One Seller and One Buyer)

$$\forall F_B \text{ FSD } F_S : \quad \text{BTR}(1, 1 + 4) \geq \text{OPT}(1, 1).$$
One Seller Stochastically Dominated by Many Buyers

Theorem (Lower Bound for One Seller and Many Buyers)

Let $M$ be any anonymous robust deterministic mechanism. For any $k$, there exist distributions $F_B$ and $F_S$ such that

$$M(1, m_B + k) < OPT(1, m_B).$$

Theorem (Upper Bound for One Seller and Many Buyers)

For any $m_B$, all distributions $F_S$ and $F_B$ satisfy

$$BTR(1, m_B + 4\sqrt{m_B}) \geq OPT(1, m_B).$$

Curiously:

$$\inf_{F_S, F_B s.t. F_B \text{ FSD } F_S} BTR(1, m_B) \rightarrow OPT(1, m_B)$$

as $m_B \to \infty.$
One Seller Stochastically Dominated by Many Buyers

Do four extra buyers suffice? Does some other constant suffice?
One Seller Stochastically Domained by Many Buyers

Do four extra buyers suffice? Does some other constant suffice?

**Theorem (Lower Bound for One Seller and Many Buyer)**

*Let* $M$ *be any anonymous robust deterministic mechanism.*

$\forall k \exists N$ s.t. $\forall m_B > N \exists$ two distributions $F_B$ FSD $F_S$ s.t.

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**Theorem**

$$ \left( \inf_{F_S, F_B \text{ s.t. } F_B \text{ FSD } F_S} \frac{\text{BTR}(1, m_B)}{\text{OPT}(1, m_B)} \right) \xrightarrow{m_B \to \infty} 1. $$
## Result Summary

Bulow-Klemperer-style results:

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Open: all gaps
I.I.D. Proof: \( \text{BTR}(m_S, m_B+1) \geq \text{OPT}(m_S, m_B) \)

- We will prove that
  \[
  \text{OPT}(m_S, m_B+1) - \text{BTR}(m_S, m_B+1) \leq \text{OPT}(m_S, m_B+1) - \text{OPT}(m_S, m_B).
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  \]
- Couple markets:
  1. Draw \( m_S + m_B + 1 \) values i.i.d. from \( F \):
     \[
     x^{(1)} \geq \ldots \geq x^{(m_S)} \geq x^{(m_S+1)} \geq \ldots \geq x^{(m_S+m_B+1)}.
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  2. Uniformly at random assign \( m_S \) as sellers, \( m_B \) as old buyers, 1 as new buyer.
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- For any \( x^{(1)} \geq \ldots \geq x^{(m_S+m_B+1)} \), we will prove in expectation over Step 2 that
  \[ E[OPT_{aug}] - E[BTR_{aug}] \leq E[OPT_{aug}] - E[OPT_{orig}]. \]
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     \]
     - \( x^{(1)} \) and \( x^{(m_S)} \) represent trading buyers and nontrading sellers.
     - \( x^{(m_S+1)} \) and \( x^{(m_S+m_B+1)} \) represent nontrading buyers and trading sellers.
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     \[
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     \( \text{trading buyers & nontrading sellers} \)
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  2. Uniformly at random assign \( m_S \) as sellers, \( m_B \) as old buyers, 1 as new buyer.
- For any \( x^{(1)} \geq \cdots \geq x^{(m_S+m_B+1)} \), we will prove in expectation over Step 2 that
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**I.I.D. Proof:** $\text{BTR}(m_S, m_B+1) \geq \text{OPT}(m_S, m_B)$

- We will prove that
  $$\text{OPT}(m_S, m_B+1) - \text{BTR}(m_S, m_B+1) \leq \text{OPT}(m_S, m_B+1) - \text{OPT}(m_S, m_B).$$
- Couple markets:
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- We will prove that
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     - Trading buyers & nontrading sellers
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**Yannai A. Gonczarowski (MSR)**
Bulow-Klemperer Results for Welfare Maximization in Two-Sided Markets
Jan 8, 2020 13 / 14
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|--------------------------|---------------------------------|-------------------------------|
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| \( \mathbb{E}[\text{OPT}_{\text{aug}}] \) | \( x^{(1)} \) \ldots \( x^{(m_S)} \) \( x^{(m_S+1)} \) \ldots \( x^{(m_S+m_B)} \) | \( \mathbb{E}[\text{OPT}_{\text{aug}}] - \mathbb{E}[\text{OPT}_{\text{orig}}] \)
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I.I.D. Proof: \( BTR(m_S, m_B+1) \geq OPT(m_S, m_B) \)

- We will prove that 
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     \]
     - \( x^{(1)} \geq \ldots \geq x^{(m_S)} \): trading buyers & nontrading sellers
     - \( x^{(m_S + 1)} \geq \ldots \geq x^{(m_S + m_B + 1)} \): nontrading buyers & trading sellers
  2. Uniformly at random assign \( m_S \) as sellers, \( m_B \) as old buyers, 1 as new buyer.

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     - Nontrading buyers and trading sellers: \( x^{(1)} \geq \cdots \geq x^{(m_S)} \geq x^{(m_S+1)} \geq \cdots \geq x^{(m_S+m_B+1)} \)

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<th>( \mathbb{E}[\text{OPT}<em>{aug}] - \mathbb{E}[\text{BTR}</em>{aug}] )</th>
<th>( \mathbb{E}[\text{OPT}<em>{aug}] - \mathbb{E}[\text{OPT}</em>{orig}] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Pr[\text{diff} \neq 0] )</td>
<td>( m_S/(m_S + m_B + 1) )</td>
<td>( m_S/(m_S + m_B + 1) )</td>
</tr>
<tr>
<td>( \mathbb{E}[\text{OPT}_{aug}] )</td>
<td>( x^{(1)} \cdots x^{(m_S)} x^{(m_S+1)} \cdots x^{(m_S+m_B)} )</td>
<td>( x^{(1)} \cdots x^{(m_S)} x^{(m_S+1)} \cdots x^{(m_S+m_B)} )</td>
</tr>
<tr>
<td>minus ...</td>
<td>( x^{(1)} \cdots x^{(\alpha)} \cdots x^{(m_S)} x^{(m_S+1)} \cdots )</td>
<td>( x^{(1)} \cdots x^{(\nu)} \cdots x^{(m_S)} x^{(m_S+1)} \cdots )</td>
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\[ \Pr[\text{diff} \neq 0] = \frac{m_S}{(m_S + m_B + 1)} \]
I.I.D. Proof: BTR\((m_S, m_B+1) \geq \text{OPT}(m_S, m_B)\)

- We will prove that
  \[
  \text{OPT}(m_S, m_B+1) - \text{BTR}(m_S, m_B+1) \leq \text{OPT}(m_S, m_B+1) - \text{OPT}(m_S, m_B).
  \]

- Couple markets:
  1. Draw \(m_S + m_B + 1\) values i.i.d. from \(F\):

  \[
  x^{(1)} \geq \ldots \geq x^{(m_S)} \geq x^{(m_S+1)} \geq \ldots \geq x^{(m_S+m_B+1)}.
  \]

  trading buyers & nontrading sellers | nontrading buyers & trading sellers

  2. Uniformly at random assign \(m_S\) as sellers, \(m_B\) as old buyers, 1 as new buyer.

- For any \(x^{(1)} \geq \ldots \geq x^{(m_S+m_B+1)}\), we will prove in expectation over Step 2 that

  \[
  \mathbb{E}[\text{OPT}_{\text{aug}}] - \mathbb{E}[\text{BTR}_{\text{aug}}] \leq \mathbb{E}[\text{OPT}_{\text{aug}}] - \mathbb{E}[\text{OPT}_{\text{orig}}].
  \]
I.I.D. Proof: \( \text{BTR}(m_S, m_B + 1) \geq \text{OPT}(m_S, m_B) \)

- We will prove that
  \[ \text{OPT}(m_S, m_B + 1) - \text{BTR}(m_S, m_B + 1) \leq \text{OPT}(m_S, m_B + 1) - \text{OPT}(m_S, m_B). \]
- Couple markets:
  1. Draw \( m_S + m_B + 1 \) values i.i.d. from \( F \):
     \[ x^{(1)} \geq \ldots \geq x^{(m_S)} \geq x^{(m_S + 1)} \geq \ldots \geq x^{(m_S + m_B + 1)}. \]
      - Trading buyers & nontrading sellers
      - Nontrading buyers & trading sellers
  2. Uniformly at random assign \( m_S \) as sellers, \( m_B \) as old buyers, 1 as new buyer.
- For any \( x^{(1)} \geq \ldots \geq x^{(m_S + m_B + 1)} \), we will prove in expectation over Step 2 that
  \[ \mathbb{E}[\text{OPT}_{\text{aug}}] - \mathbb{E}[\text{BTR}_{\text{aug}}] \leq \mathbb{E}[\text{OPT}_{\text{aug}}] - \mathbb{E}[\text{OPT}_{\text{orig}}]. \]

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<td>Pr[diff ( \neq 0 )]</td>
<td>( \frac{m_S}{m_S + m_B + 1} )</td>
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<tr>
<td>( \mathbb{E}[\text{OPT}_{\text{aug}}] )</td>
<td>( x^{(1)} \ldots x^{(m_S)} x^{(m_S + 1)} \ldots x^{(m_S + m_B)} )</td>
<td>( x^{(1)} \ldots x^{(m_S)} x^{(m_S + 1)} \ldots x^{(m_S + m_B)} )</td>
</tr>
<tr>
<td>minus ...</td>
<td>( x^{(1)} \ldots x^{(\alpha)} x^{(m_S)} x^{(m_S + 1)} \ldots ) ( \downarrow ) buyer q</td>
<td>( x^{(1)} \ldots x^{(\nu)} \ldots x^{(m_S)} x^{(m_S + 1)} \ldots ) ( \downarrow ) new buyer</td>
</tr>
<tr>
<td>diff + ( x^{(m_S + 1)} )</td>
<td>min of ( q \geq 1 ) vals ( \sim U({x^{(1)}, \ldots , x^{(m_S)}}) )</td>
<td>val ( \sim U({x^{(1)}, \ldots , x^{(m_S)}}) )</td>
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I.I.D. Proof: \( \text{BTR}(m_S, m_B + 1) \geq \text{OPT}(m_S, m_B) \)

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  \]
Questions?

Thank you!

"Sorry, no trades. Cash only."