Sisterhood in the Gale-Shapley Matching Algorithm

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The Stable Matching Problem

- Two disjoint finite sets to be matched: women W and men M.
  - Assume 1-to-1 for now.
  - Assume $|W| = |M|$ for now.

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  - Assume $|W| = |M|$ for now.
- Preferences for each woman and for each man.
  - Assume a strict order of preference for each woman over all men and vice versa.
  - Assume no blacklists for now.
- The goal: a stable matching.
  - If w and m are matched, and if w' and m' are matched, then w and m' should not both prefer each other over their spouses.

The Gale-Shapley Deferred-Acceptance Algorithm

Gale and Shapley (1962)

The following algorithm yields a stable matching.

1. On each night, every man serenades under the window of the woman he prefers most out of those who have not yet rejected him.

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3. When no more rejections occur, each woman is matched with the man serenading under her window.

Gender Duality and Manipulation Incentives

Gale and Shapley (1962)

No stable matching is better for any man.

McVitie and Wilson (1971)

No stable matching is worse for any woman.

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No subset of men can lie in a way that would make them all better off lying.

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Generally, there exists a woman who would be better off lying.

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Note: the latter two do not follow from the former two.

Example: Manipulation by Women

Men's Preferences

m_1	w_2	w_1	w_3
m_2	w_1	w_2	w_3
m_3	w_1	w_3	w_2

Women's Preferences

w_1	$m_1 > m_2 > m_3$
w_2	$m_2 > m_1$
w_3	any

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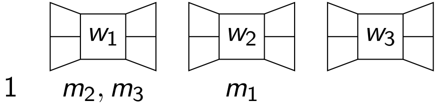

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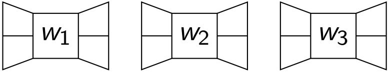

	w_1	w_2	w_3
1	m_2, m_3	m_1	
2	m_2	m_1	m_3

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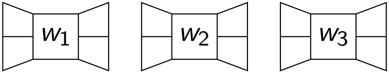

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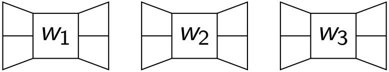

	w_1	w_2	w_3
1	m_2, m_3	m_1	
2	m_2	m_1	m_3
2	m_3	m_1, m_2	

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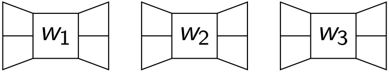

	w_1	w_2	w_3
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2	m_2	m_1	m_3
2	m_3	m_1, m_2	
3	m_1, m_3	m_2	

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4	m_1	m_2	m_3

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- w_1 improved her match, but so did w_2; and w_3 is unharmed.

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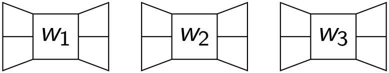

	w_1	w_2	w_3
1	m_2, m_3	m_1	
2	m_2	m_1	m_3
2	m_3	m_1, m_2	
3	m_1, m_3	m_2	
4	m_1	m_2	m_3

- w_1 improved her match, but so did w_2; and w_3 is unharmed.
- w_1 made w_2 "give up" m_1 by making sure w_2 is approached by someone w_2 prefers better.

Sisterhood Theorem

Assume that a subset of the women declare false orders of preference for themselves.

We examines two runs of the Gale-Shapley algorithm:

- OA — according to everyone's true preferences; yields the matching O.
- NA — according to the liars' false preferences, and everyone else's true preferences; yields the matching N.

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Theorem (Sisterhood)

Under the above conditions, if all lying women are weakly better off, then:

1. *All women are weakly better off.*
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No such "hoodness" exists within any other subset of $W \cup M$. Indeed, when even a single man lies and is weakly b/o, some women and men may be b/o, and some others — w/o.

An Easy Proof?

Observation

If every lying woman w lies in an optimal way (i.e. the lies constitute a Nash Equilibrium in the lying game), then the new matching is stable.

Background

Monogamous Case

Polygamy, Blacklists, and Mismatched Quotas

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Proof.

The new matching is obviously stable w.r.t. the new preferences. It is thus enough to consider couples in which at least one liar participates.

$$\begin{array}{cc} w & w' \\ | & | \\ m & m' \end{array}$$

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So, what's the problem? Why would someone lie in a non-optimal way? Why do we care about non-equilibrium?

When a Lie Need Not be Optimal

Men's Preferences

m_1	w_1	w_3	w_2	w_4
m_2	w_2	w_3	any	any
m_3	w_3	w_2	w_1	w_4
m_4	w_1	w_4	any	any

Women's Preferences

w_1	$m_3 > m_1 > m_2, m_4$
w_2	$m_3 > m_1 > m_2, m_4$
w_3	$m_2 > m_1 > m_3$
w_4	any

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m_2	w_2	w_3	any	any
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m_4	w_1	w_4	any	any

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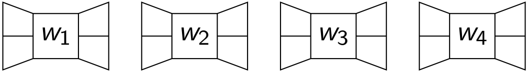

1 m_4, m_1 m_2 m_3

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m_1	w_1	w_3	w_2	w_4
m_2	w_2	w_3	any	any
m_3	w_3	w_2	w_1	w_4
m_4	w_1	w_4	any	any

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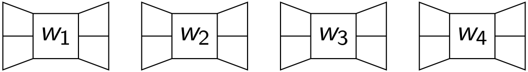

	w_1	w_2	w_3	w_4
1	m_4, m_1	m_2	m_3	
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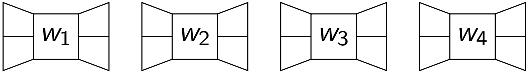
Women's Preferences

w_1	$m_3 > m_{\not{1}4} > m_2, m_{\not{4}1}$
w_2	$m_3 > m_1 > m_2, m_4$
w_3	$m_2 > m_1 > m_3$
w_4	any



	w_1	w_2	w_3	w_4
1	m_4, m_1	m_2	m_3	
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2	m_1	m_2	m_3	m_4
2	m_4	m_2	m_1, m_3	

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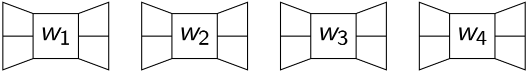

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1	m_4, m_1	m_2	m_3	
2	m_1	m_2	m_3	m_4
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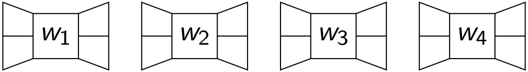

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1	m_4, m_1	m_2	m_3	
2	m_1	m_2	m_3	m_4
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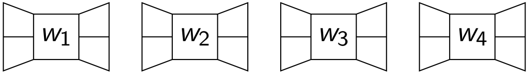

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2	m_4	m_2	m_1, m_3	
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4	m_4	m_3	m_1, m_2	
5	m_4	m_1, m_3	m_2	
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m_4	w_1	w_4	any	any

Women's Preferences

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w_4	any

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1	m_4, m_1	m_2	m_3	
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3	m_4	m_2, m_3	m_1	
4	m_4	m_3	m_1, m_2	
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5	m_4	m_1, m_3	m_2	
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In this example:

- The truth is an optimal strategy for any coalition not including w_1.

When a Lie Need Not be Optimal (cont.)

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- The truth is an optimal strategy for any coalition not including w_1.
- No strategy for w_1 is better than the truth if all other women respond optimally to it.

When a Lie Need Not be Optimal (cont.)

In this example:

- The truth is an optimal strategy for any coalition not including w_1.
- No strategy for w_1 is better than the truth if all other women respond optimally to it.
- Thus, in no Nash equilibrium is any woman better-matched than according to all the true preferences.

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- w_2's lie in this strategy is not equivalent to any truncation of her preferences.

One/Many-To-Many Matchings and Blacklists

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- We still assume total preferences over individuals.
- For a person p, denote $O(p) = (o_1^p, \ldots, o_{|O(p)|}^p)$ and $N(p) = (n_1^p, \ldots, n_{|N(p)|}^p)$. Lower index = higher on p's list.

Sisterhood Theorem - Polygamous Case

Definition (Improvement)

A woman p is said to be *weakly better off* if:

1. $N(p)$ contains no-one who is blacklisted by p.
2. $|N(p)| \geq |O(p)|$.
3. For each $1 \leq i \leq O(p)$, p weakly prefers n_i^p over o_i^p.

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Theorem (Sisterhood)

If all lying women are weakly better off, then:

1. *All women are weakly better off.*
2. *All men have gained only worse matches.*

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A Rural-Hospitals-type Theorem

Under the above conditions,

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Corollary

If all women have the same order of preference, then under the above conditions the matching must remain unchanged. Therefore, in this case there is no "significant" incentive for any subset of women to lie, even for the sake of one of them.

Questions?

Thank you!