Manipulation of Stable Matchings using Minimal Blacklists

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Roth (2002)

"Successful matching mechanisms produce stable outcomes."

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Gale and Sotomayor (1985)

Generally, there is a woman who would be better off lying when the M-optimal stable matching is used.

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Gusfield and Irving (1989)

No results are known regarding achieving this by any means other than such preference-list truncation, i.e. by also permuting preference lists.

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Define $n \triangleq |W| = |M|$. The women may force the W-optimal stable matching as the M-optimal one, using a profile of preference lists with average blacklist size no more than ...

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2. $O(\log n)$
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5. $\frac{n}{c}$
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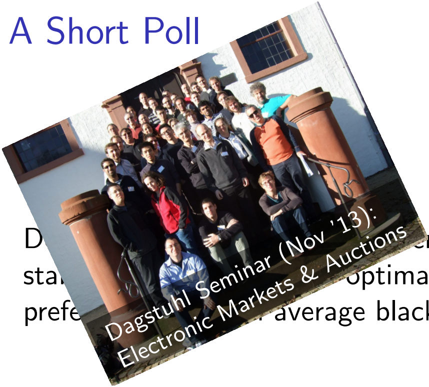

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Answering Gusfield and Irving's Open Question

Summary of Main Result (Weak Version)

- The women may force any M-rational perfect matching as the unique stable matching, using a profile of preference lists in which at most half of the women have blacklists, and in which the average blacklist size is less than 1.

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If women pay a price for every man they blacklist, then order-of-magnitude improvement.

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- (*cf.* the shoe market.)
- Completely different proofs.

Improved Insight into Matching Markets

Both phase-change results lead to a similar conclusion in different senses:

The preferences of the smaller side of the market (even if only slightly smaller) play a far more significant role than may be expected in determining the stable matchings, and those of the larger side — a considerably insignificant one.

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More generally: our results shed light on the question of how much, if at all, do given preferences for one side *a priori* impose limitations on the set of stable matchings under various conditions.

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  single lottery = random serial (buyer) dictatorship
  $\Rightarrow$ Pareto-efficient outcome.

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Define $n \triangleq |W| = |M|$. Let $\mathcal{P}_M$ be a profile of preference lists for M. For every M-rational perfect matching μ, there exists a profile $\mathcal{P}_W$ of preference lists for W, s.t. all the following hold.

1. *The unique stable matching, given $\mathcal{P}_W$ and $\mathcal{P}_M$, is μ.*
2. *The blacklists in $\mathcal{P}_W$ are pairwise disjoint, i.e. no man appears in more than one blacklist.*
3. *n_b, the number of women who have nonempty blacklists in $\mathcal{P}_W$, is at most $\frac{n}{2}$.*
4. *The combined size of all blacklists in $\mathcal{P}_W$ is at most $n - n_b$, i.e. at most the number of women who have empty blacklists.*

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Furthermore, $\mathcal{P}_W$ can be computed in worst-case $O(n^3)$ time, best-case $O(n^2)$ time and average-case (assuming μ is uniformly distributed given $\mathcal{P}_M$) $O(n^2 \log n)$ time.

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The following algorithm yields the M-optimal stable matching.

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1. Setup: Every man serenades under the window of the woman he prefers most.

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The following algorithm yields the M-optimal stable matching.

1. Setup: Every man serenades under the window of the woman he prefers most.
2. A man is *scheduled for rejection* if he is blacklisted by the woman to whom he serenades, or if she prefers another man currently serenading under her window.

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2. A man is *scheduled for rejection* if he is blacklisted by the woman to whom he serenades, or if she prefers another man currently serenading under her window.
3. On each *night*, choose an arbitrary man scheduled for rejection. He moves to serenade under the window of the woman next on his preference list, if such woman exists.

The (unique) M-optimal matching is always reached, regardless of the arbitrary choices made during the run.

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3. On each *night*, choose an arbitrary man scheduled for rejection. He moves to serenade under the window of the woman next on his preference list, if such woman exists.
4. When no men are scheduled for rejection, the algorithm terminates. Each woman is matched with the man serenading under her window; everyone else is unmatched.

The (unique) M-optimal matching is always reached, regardless of the arbitrary choices made during the run.

Tightness Overview

m_1	$w_2 > w_3 > w_4 > w_1$
m_2	$w_3 > w_4 > w_1 > w_2$
m_3	$w_4 > w_1 > w_2 > w_3$
m_4	$w_1 > w_2 > w_3 > w_4$

Tightness Overview

m_1	$w_2 > w_3 > w_4 > w_1$
m_2	$w_3 > w_4 > w_1 > w_2$
m_3	$w_4 > w_1 > w_2 > w_3$
m_4	$w_1 > w_2 > w_3 > w_4$

Tightness Overview

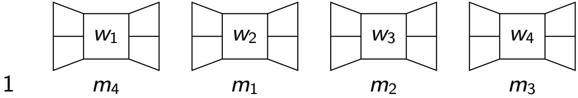
m_1	$w_2 > w_3 > w_4 > w_1$
m_2	$w_3 > w_4 > w_1 > w_2$
m_3	$w_4 > w_1 > w_2 > w_3$
m_4	$w_1 > w_2 > w_3 > w_4$

w_1	m_1 (Blacklist: m_4, m_3, m_2)
w_2	$m_2 > m_1 > m_4 > m_3$
w_3	$m_3 > m_2 > m_1 > m_4$
w_4	$m_4 > m_3 > m_2 > m_1$

Tightness Overview

m_1	$w_2 > w_3 > w_4 > w_1$
m_2	$w_3 > w_4 > w_1 > w_2$
m_3	$w_4 > w_1 > w_2 > w_3$
m_4	$w_1 > w_2 > w_3 > w_4$

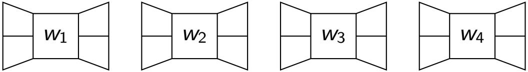
w_1	m_1 (Blacklist: m_4, m_3, m_2)
w_2	$m_2 > m_1 > m_4 > m_3$
w_3	$m_3 > m_2 > m_1 > m_4$
w_4	$m_4 > m_3 > m_2 > m_1$



Tightness Overview

m_1	$w_2 > w_3 > w_4 > w_1$
m_2	$w_3 > w_4 > w_1 > w_2$
m_3	$w_4 > w_1 > w_2 > w_3$
m_4	$w_1 > w_2 > w_3 > w_4$

w_1	m_1 (Blacklist: m_4, m_3, m_2)
w_2	$m_2 > m_1 > m_4 > m_3$
w_3	$m_3 > m_2 > m_1 > m_4$
w_4	$m_4 > m_3 > m_2 > m_1$

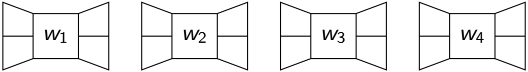


1	m_4	m_1	m_2	m_3
2		m_1, m_4	m_2	m_3

Tightness Overview

m_1	$w_2 > w_3 > w_4 > w_1$
m_2	$w_3 > w_4 > w_1 > w_2$
m_3	$w_4 > w_1 > w_2 > w_3$
m_4	$w_1 > w_2 > w_3 > w_4$

w_1	m_1 (Blacklist: m_4, m_3, m_2)
w_2	$m_2 > m_1 > m_4 > m_3$
w_3	$m_3 > m_2 > m_1 > m_4$
w_4	$m_4 > m_3 > m_2 > m_1$

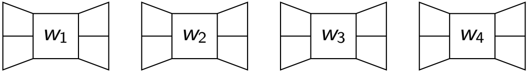


	w_1	w_2	w_3	w_4
1	m_4	m_1	m_2	m_3
2		m_1, m_4	m_2	m_3
3		m_1	m_2, m_4	m_3

Tightness Overview

m_1	$w_2 > w_3 > w_4 > w_1$
m_2	$w_3 > w_4 > w_1 > w_2$
m_3	$w_4 > w_1 > w_2 > w_3$
m_4	$w_1 > w_2 > w_3 > w_4$

w_1	m_1 (Blacklist: m_4, m_3, m_2)
w_2	$m_2 > m_1 > m_4 > m_3$
w_3	$m_3 > m_2 > m_1 > m_4$
w_4	$m_4 > m_3 > m_2 > m_1$



	w_1	w_2	w_3	w_4
1	m_4	m_1	m_2	m_3
2		m_1, m_4	m_2	m_3
3		m_1	m_2, m_4	m_3
4		m_1	m_2	m_3, m_4

Tightness Overview

m_1	$w_2 > w_3 > w_4 > w_1$
m_2	$w_3 > w_4 > w_1 > w_2$
m_3	$w_4 > w_1 > w_2 > w_3$
m_4	$w_1 > w_2 > w_3 > w_4$

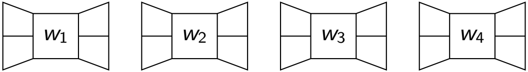
w_1	m_1 (Blacklist: m_4, m_3, m_2)
w_2	$m_2 > m_1 > m_4 > m_3$
w_3	$m_3 > m_2 > m_1 > m_4$
w_4	$m_4 > m_3 > m_2 > m_1$

	w_1	w_2	w_3	w_4
1	m_4	m_1	m_2	m_3
2		m_1, m_4	m_2	m_3
3		m_1	m_2, m_4	m_3
4		m_1	m_2	m_3, m_4
5	m_3	m_1	m_2	m_4

Tightness Overview

m_1	$w_2 > w_3 > w_4 > w_1$
m_2	$w_3 > w_4 > w_1 > w_2$
m_3	$w_4 > w_1 > w_2 > w_3$
m_4	$w_1 > w_2 > w_3 > w_4$

w_1	m_1 (Blacklist: m_4, m_3, m_2)
w_2	$m_2 > m_1 > m_4 > m_3$
w_3	$m_3 > m_2 > m_1 > m_4$
w_4	$m_4 > m_3 > m_2 > m_1$

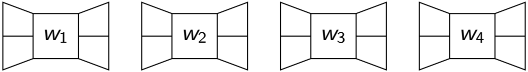


	w_1	w_2	w_3	w_4
1	m_4	m_1	m_2	m_3
2		m_1, m_4	m_2	m_3
3		m_1	m_2, m_4	m_3
4		m_1	m_2	m_3, m_4
5	m_3	m_1	m_2	m_4

Tightness Overview

m_1	$w_2 > w_3 > w_4 > w_1$
m_2	$w_3 > w_4 > w_1 > w_2$
m_3	$w_4 > w_1 > w_2 > w_3$
m_4	$w_1 > w_2 > w_3 > w_4$

w_1	m_1 (Blacklist: m_4, m_3, m_2)
w_2	$m_2 > m_1 > m_4 > m_3$
w_3	$m_3 > m_2 > m_1 > m_4$
w_4	$m_4 > m_3 > m_2 > m_1$



	w_1	w_2	w_3	w_4
1	m_4	m_1	m_2	m_3
2		m_1 , m_4	m_2	m_3
3		m_1	m_2 , m_4	m_3
4		m_1	m_2	m_3 , m_4
5	m_3	m_1	m_2	m_4
6		m_1 , m_3	m_2	m_4

Tightness Overview

m_1	$w_2 > w_3 > w_4 > w_1$
m_2	$w_3 > w_4 > w_1 > w_2$
m_3	$w_4 > w_1 > w_2 > w_3$
m_4	$w_1 > w_2 > w_3 > w_4$

w_1	m_1 (Blacklist: m_4, m_3, m_2)
w_2	$m_2 > m_1 > m_4 > m_3$
w_3	$m_3 > m_2 > m_1 > m_4$
w_4	$m_4 > m_3 > m_2 > m_1$

	w_1	w_2	w_3	w_4
1	m_4	m_1	m_2	m_3
2		m_1, m_4	m_2	m_3
3		m_1	m_2, m_4	m_3
4		m_1	m_2	m_3, m_4
5	m_3	m_1	m_2	m_4
6		m_1, m_3	m_2	m_4
7		m_1	m_2, m_3	m_4

Tightness Overview

m_1	$w_2 > w_3 > w_4 > w_1$
m_2	$w_3 > w_4 > w_1 > w_2$
m_3	$w_4 > w_1 > w_2 > w_3$
m_4	$w_1 > w_2 > w_3 > w_4$

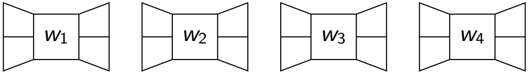
w_1	m_1 (Blacklist: m_4, m_3, m_2)
w_2	$m_2 > m_1 > m_4 > m_3$
w_3	$m_3 > m_2 > m_1 > m_4$
w_4	$m_4 > m_3 > m_2 > m_1$

	w_1	w_2	w_3	w_4
1	m_4	m_1	m_2	m_3
2		m_1, m_4	m_2	m_3
3		m_1	m_2, m_4	m_3
4		m_1	m_2	m_3, m_4
5	m_3	m_1	m_2	m_4
6		m_1, m_3	m_2	m_4
7		m_1	m_2, m_3	m_4
8		m_1	m_3	m_4, m_2

Tightness Overview

m_1	$w_2 > w_3 > w_4 > w_1$
m_2	$w_3 > w_4 > w_1 > w_2$
m_3	$w_4 > w_1 > w_2 > w_3$
m_4	$w_1 > w_2 > w_3 > w_4$

w_1	m_1 (Blacklist: m_4, m_3, m_2)
w_2	$m_2 > m_1 > m_4 > m_3$
w_3	$m_3 > m_2 > m_1 > m_4$
w_4	$m_4 > m_3 > m_2 > m_1$

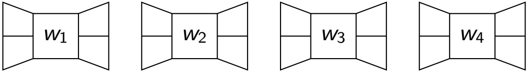


	w_1	w_2	w_3	w_4
1	m_4	m_1	m_2	m_3
2		m_1, m_4	m_2	m_3
3		m_1	m_2, m_4	m_3
4		m_1	m_2	m_3, m_4
5	m_3	m_1	m_2	m_4
6		m_1, m_3	m_2	m_4
7		m_1	m_2, m_3	m_4
8		m_1	m_3	m_4, m_2
9	m_2	m_1	m_3	m_4

Tightness Overview

m_1	$w_2 > w_3 > w_4 > w_1$
m_2	$w_3 > w_4 > w_1 > w_2$
m_3	$w_4 > w_1 > w_2 > w_3$
m_4	$w_1 > w_2 > w_3 > w_4$

w_1	m_1 (Blacklist: m_4, m_3, m_2)
w_2	$m_2 > m_1 > m_4 > m_3$
w_3	$m_3 > m_2 > m_1 > m_4$
w_4	$m_4 > m_3 > m_2 > m_1$

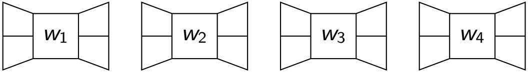


	w_1	w_2	w_3	w_4
1	m_4	m_1	m_2	m_3
2		m_1, m_4	m_2	m_3
3		m_1	m_2, m_4	m_3
4		m_1	m_2	m_3, m_4
5	m_3	m_1	m_2	m_4
6		m_1, m_3	m_2	m_4
7		m_1	m_2, m_3	m_4
8		m_1	m_3	m_4, m_2
9	m_2	m_1	m_3	m_4

Tightness Overview

m_1	$w_2 > w_3 > w_4 > w_1$
m_2	$w_3 > w_4 > w_1 > w_2$
m_3	$w_4 > w_1 > w_2 > w_3$
m_4	$w_1 > w_2 > w_3 > w_4$

w_1	m_1 (Blacklist: m_4, m_3, m_2)
w_2	$m_2 > m_1 > m_4 > m_3$
w_3	$m_3 > m_2 > m_1 > m_4$
w_4	$m_4 > m_3 > m_2 > m_1$

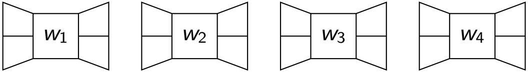


	w_1	w_2	w_3	w_4
1	m_4	m_1	m_2	m_3
2		m_1, m_4	m_2	m_3
3		m_1	m_2, m_4	m_3
4		m_1	m_2	m_3, m_4
5	m_3	m_1	m_2	m_4
6		m_1, m_3	m_2	m_4
7		m_1	m_2, m_3	m_4
8		m_1	m_3	m_4, m_2
9	m_2	m_1	m_3	m_4
10		m_1, m_2	m_3	m_4

Tightness Overview

m_1	$w_2 > w_3 > w_4 > w_1$
m_2	$w_3 > w_4 > w_1 > w_2$
m_3	$w_4 > w_1 > w_2 > w_3$
m_4	$w_1 > w_2 > w_3 > w_4$

w_1	m_1 (Blacklist: m_4, m_3, m_2)
w_2	$m_2 > m_1 > m_4 > m_3$
w_3	$m_3 > m_2 > m_1 > m_4$
w_4	$m_4 > m_3 > m_2 > m_1$



	w_1	w_2	w_3	w_4
1	m_4	m_1	m_2	m_3
2		m_1, m_4	m_2	m_3
3		m_1	m_2, m_4	m_3
4		m_1	m_2	m_3, m_4
5	m_3	m_1	m_2	m_4
6		m_1, m_3	m_2	m_4
7		m_1	m_2, m_3	m_4
8		m_1	m_3	m_4, m_2
9	m_2	m_1	m_3	m_4
10		m_1, m_2	m_3	m_4
11		m_2	m_3, m_1	m_4

Tightness Overview

m_1	$w_2 > w_3 > w_4 > w_1$
m_2	$w_3 > w_4 > w_1 > w_2$
m_3	$w_4 > w_1 > w_2 > w_3$
m_4	$w_1 > w_2 > w_3 > w_4$

w_1	m_1 (Blacklist: m_4, m_3, m_2)
w_2	$m_2 > m_1 > m_4 > m_3$
w_3	$m_3 > m_2 > m_1 > m_4$
w_4	$m_4 > m_3 > m_2 > m_1$

	w_1	w_2	w_3	w_4
1	m_4	m_1	m_2	m_3
2		m_1, m_4	m_2	m_3
3		m_1	m_2, m_4	m_3
4		m_1	m_2	m_3, m_4
5	m_3	m_1	m_2	m_4
6		m_1, m_3	m_2	m_4
7		m_1	m_2, m_3	m_4
8		m_1	m_3	m_4, m_2
9	m_2	m_1	m_3	m_4
10		m_1, m_2	m_3	m_4
11		m_2	m_3, m_1	m_4
12		m_2	m_3	m_4, m_1

Tightness Overview

m_1	$w_2 > w_3 > w_4 > w_1$
m_2	$w_3 > w_4 > w_1 > w_2$
m_3	$w_4 > w_1 > w_2 > w_3$
m_4	$w_1 > w_2 > w_3 > w_4$

w_1	m_1 (Blacklist: m_4, m_3, m_2)
w_2	$m_2 > m_1 > m_4 > m_3$
w_3	$m_3 > m_2 > m_1 > m_4$
w_4	$m_4 > m_3 > m_2 > m_1$

	w_1	w_2	w_3	w_4
1	m_4	m_1	m_2	m_3
2		m_1, m_4	m_2	m_3
3		m_1	m_2, m_4	m_3
4		m_1	m_2	m_3, m_4
5	m_3	m_1	m_2	m_4
6		m_1, m_3	m_2	m_4
7		m_1	m_2, m_3	m_4
8		m_1	m_3	m_4, m_2
9	m_2	m_1	m_3	m_4
10		m_1, m_2	m_3	m_4
11		m_2	m_3, m_1	m_4
12		m_2	m_3	m_4, m_1
13	m_1	m_2	m_3	m_4

Tightness Overview

m_1	$w_2 > w_3 > w_4 > w_1$
m_2	$w_3 > w_4 > w_1 > w_2$
m_3	$w_4 > w_1 > w_2 > w_3$
m_4	$w_1 > w_2 > w_3 > w_4$

w_1	m_1 (Blacklist: m_4, m_3, m_2)
w_2	$m_2 > m_1 > m_4 > m_3$
w_3	$m_3 > m_2 > m_1 > m_4$
w_4	$m_4 > m_3 > m_2 > m_1$

	w_1	w_2	w_3	w_4
1	m_4	m_1	m_2	m_3
2		m_1, m_4	m_2	m_3
3		m_1	m_2, m_4	m_3
4		m_1	m_2	m_3, m_4
5	m_3	m_1	m_2	m_4
6		m_1, m_3	m_2	m_4
7		m_1	m_2, m_3	m_4
8		m_1	m_3	m_4, m_2
9	m_2	m_1	m_3	m_4
10		m_1, m_2	m_3	m_4
11		m_2	m_3, m_1	m_4
12		m_2	m_3	m_4, m_1
13	m_1	m_2	m_3	m_4

Tightness Overview

m_1	$w_2 > w_3 > w_4 > w_1$
m_2	$w_3 > w_4 > w_1 > w_2$
m_3	$w_4 > w_1 > w_2 > w_3$
m_4	$w_1 > w_2 > w_3 > w_4$

w_1	m_1 (Blacklist: m_4, m_3, m_2)
w_2	$m_2 > m_1 > m_4 > m_3$
w_3	$m_3 > m_2 > m_1 > m_4$
w_4	$m_4 > m_3 > m_2 > m_1$

	w_1	w_2	w_3	w_4
1	m_4	m_1	m_2	m_3
2		m_*, m_*	m_2	m_3
3		m_*	m_*, m_*	m_3
4		m_*	m_*	m_*, m_*
5	m_*	m_*	m_*	m_*
6		m_*, m_*	m_*	m_*
7		m_*	m_*, m_*	m_*
8		m_*	m_*	m_*, m_*
9	m_*	m_*	m_*	m_*
10		m_*, m_2	m_*	m_*
11		m_2	m_3, m_*	m_*
12		m_2	m_3	m_4, m_1
13	m_1	m_2	m_3	m_4

Tightness Overview

m_1	$w_2 > w_3 > w_4 > w_1$
m_2	$w_3 > w_4 > w_1 > w_2$
m_3	$w_4 > w_1 > w_2 > w_3$
m_4	$w_1 > w_2 > w_3 > w_4$

w_1	m_1 (Blacklist: m_4, m_3, m_2)
w_2	$m_2 > m_1 > m_4 > m_3$
w_3	$m_3 > m_2 > m_1 > m_4$
w_4	$m_4 > m_3 > m_2 > m_1$

	w_1	w_2	w_3	w_4
1	m_4	m_1	m_2	m_3
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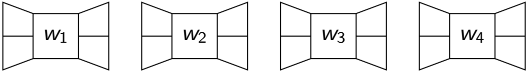

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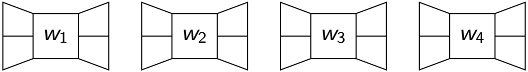

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- When unmatched men exist, we're back to $\Theta(n^2)$.* General idea: follow the naïve construction; use these men as "placeholders" to initiate cycles without blacklisting.

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- See the full paper (on arXiv) for the full results.

Questions?

Thank you!