Timely Common Knowledge

Characterising Asymmetric Distributed Coordination via Vectorial Fixed Points

Yannai A. Gonczarowski

Einstein Institute of Mathematics and Center for the Study of Rationality
The Hebrew University of Jerusalem

January 8, 2013

Joint work with Yoram Moses

A Motivational Example

- Consider a robotic Car Wash.

A Motivational Example

- Consider a robotic Car Wash.
  - Two washing robots: L and R.
  - A drying robot D.

A Motivational Example

- Consider a robotic Car Wash.
  - Two washing robots: L and R.
  - A drying robot D.
- At some point after a car enters, it must be soaped & rinsed from both sides, and then dried.

A Motivational Example

- Consider a robotic Car Wash.
  - Two washing robots: L and R.
  - A drying robot D.
- At some point after a car enters, it must be soaped & rinsed from both sides, and then dried.
  - L is of a new model — takes 4 minutes to wash.
  - R is older — takes 6 minutes.

A Motivational Example

- Consider a robotic Car Wash.
  - Two washing robots: L and R.
  - A drying robot D.
- At some point after a car enters, it must be soaped & rinsed from both sides, and then dried.
  - L is of a new model — takes 4 minutes to wash.
  - R is older — takes 6 minutes.
  - Drying should not begin more than 5 minutes after the first washing robot finished.

A Motivational Example

- Consider a robotic Car Wash.
  - Two washing robots: L and R.
  - A drying robot D.
- At some point after a car enters, it must be soaped & rinsed from both sides, and then dried.
  - L is of a new model — takes 4 minutes to wash.
  - R is older — takes 6 minutes.
  - Drying should not begin more than 5 minutes after the first washing robot finished.
- A few immediate consequences:

A Motivational Example

- Consider a robotic Car Wash.
  - Two washing robots: L and R.
  - A drying robot D.
- At some point after a car enters, it must be soaped & rinsed from both sides, and then dried.
  - L is of a new model — takes 4 minutes to wash.
  - R is older — takes 6 minutes.
  - Drying should not begin more than 5 minutes after the first washing robot finished.
- A few immediate consequences:
  - L and R may not finish more than 5 minutes apart.

A Motivational Example

- Consider a robotic Car Wash.
  - Two washing robots: L and R.
  - A drying robot D.
- At some point after a car enters, it must be soaped & rinsed from both sides, and then dried.
  - L is of a new model — takes 4 minutes to wash.
  - R is older — takes 6 minutes.
  - Drying should not begin more than 5 minutes after the first washing robot finished.
- A few immediate consequences:
  - L and R may not finish more than 5 minutes apart.
  - R must start washing no more than 3 minutes after, and no more than 7 minutes before L starts.

A Motivational Example

- Consider a robotic Car Wash.
  - Two washing robots: L and R.
  - A drying robot D.
- At some point after a car enters, it must be soaped & rinsed from both sides, and then dried.
  - L is of a new model — takes 4 minutes to wash.
  - R is older — takes 6 minutes.
  - Drying should not begin more than 5 minutes after the first washing robot finished.
- A few immediate consequences:
  - L and R may not finish more than 5 minutes apart.
  - R must start washing no more than 3 minutes after, and no more than 7 minutes before L starts.
  - ...

Primary Goal

Constraints

$$\begin{array}{rcccl} -7 & \leq & \text{start}_r - \text{start}_l & \leq & 3 \\ 6 & \leq & \text{start}_d - \text{start}_r & \leq & 11 \\ -9 & \leq & \text{start}_l - \text{start}_d & \leq & -4 \end{array}$$

- How can the tasks be coordinated?

Primary Goal

Constraints

$$\begin{aligned} -7 &\leq \text{start}_r - \text{start}_l \leq 3 \\ 6 &\leq \text{start}_d - \text{start}_r \leq 11 \\ -9 &\leq \text{start}_l - \text{start}_d \leq -4 \end{aligned}$$

- How can the tasks be coordinated?
    - What if the robots cannot see each other / the car?

Primary Goal

Constraints

$$
\begin{array}{rcrcl}
-7 & \leq & \mathsf{start}_r - \mathsf{start}_l & \leq & 3 \\
6 & \leq & \mathsf{start}_d - \mathsf{start}_r & \leq & 11 \\
-9 & \leq & \mathsf{start}_l - \mathsf{start}_d & \leq & -4
\end{array}
$$

- How can the tasks be coordinated?
  - What if the robots cannot see each other / the car?
  - What if messages may take a while to arrive?

Primary Goal

Constraints

$$
\begin{array}{rcccr}
-7 & \leq & \mathrm{start}_r - \mathrm{start}_l & \leq & 3 \\
6 & \leq & \mathrm{start}_d - \mathrm{start}_r & \leq & 11 \\
-9 & \leq & \mathrm{start}_l - \mathrm{start}_d & \leq & -4
\end{array}
$$

- How can the tasks be coordinated?
  - What if the robots cannot see each other / the car?
  - What if messages may take a while to arrive?
  - What if messages may never arrive?

Primary Goal

Constraints

$$\begin{array}{rcrcr} -7 & \leq & start_r - start_l & \leq & 3 \\ 6 & \leq & start_d - start_r & \leq & 11 \\ -9 & \leq & start_l - start_d & \leq & -4 \end{array}$$

- How can the tasks be optimally coordinated?
  - What if the robots cannot see each other / the car?
  - What if messages may take a while to arrive?
  - What if messages may never arrive?

Primary Goal

Constraints

$$
\begin{array}{rcccr}
-7 & \leq & start_r - start_l & \leq & 3 \\
6 & \leq & start_d - start_r & \leq & 11 \\
-9 & \leq & start_l - start_d & \leq & -4
\end{array}
$$

- How can the tasks be optimally coordinated?
  - What if the robots cannot see each other / the car?
  - What if messages may take a while to arrive?
  - What if messages may never arrive?

Epistemic Consequences

Epistemic Consequences

Knowledge of Precondition

If a protocol P guaranteeing that L starts to wash only if a given event ψ holds, then whenever L starts to wash (according to P), L knows that ψ holds.

Epistemic Consequences

Knowledge of Precondition

If a protocol P guaranteeing that L starts to wash only if a given event ψ holds, then whenever L starts to wash (according to P), L knows that ψ holds.

- When L acts, it must know that the car has arrived.

Epistemic Consequences

Knowledge of Precondition

If a protocol P guaranteeing that L starts to wash only if a given event ψ holds, then whenever L starts to wash (according to P), L knows that ψ holds.

- When L acts, it must know that the car has arrived.
- It must also know that between 7 time units earlier and 3 time units later, R knows that the car has already arrived.

Epistemic Consequences

Knowledge of Precondition

If a protocol P guaranteeing that L starts to wash only if a given event ψ holds, then whenever L starts to wash (according to P), L knows that ψ holds.

- When L acts, it must know that the car has arrived.
- It must also know that between 7 time units earlier and 3 time units later, R knows that the car has already arrived.
- It must also know that between 7 time units earlier and 3 time units later, R knows that between 3 time units earlier and 7 time units later, L knows that the car has already arrived.

Epistemic Consequences

Knowledge of Precondition

If a protocol P guaranteeing that L starts to wash only if a given event ψ holds, then whenever L starts to wash (according to P), L knows that ψ holds.

- When L acts, it must know that the car has arrived.
- It must also know that between 7 time units earlier and 3 time units later, R knows that the car has already arrived.
- It must also know that between 7 time units earlier and 3 time units later, R knows that between 3 time units earlier and 7 time units later, L knows that the car has already arrived.
- etc.

Epistemic Consequences

Knowledge of Precondition

If a protocol P guaranteeing that L starts to wash only if a given event ψ holds, then whenever L starts to wash (according to P), L knows that ψ holds.

- When L acts, it must know that the car has arrived.
- It must also know that between 7 time units earlier and 3 time units later, R knows that the car has already arrived.
- It must also know that between 7 time units earlier and 3 time units later, R knows that between 3 time units earlier and 7 time units later, L knows that the car has already arrived.
- etc.
- Codependence $\Rightarrow$ infinitely many epistemic requirements.

Review: Knowledge Gain and Coordination

Many forms of coordination have a necessary and sufficient characterising epistemic state.

Review: Knowledge Gain and Coordination

Many forms of coordination have a necessary and sufficient characterising epistemic state.

Halpern and Moses (1990): Common knowledge is necessary and sufficient for simultaneous actions.

Review: Knowledge Gain and Coordination

Many forms of coordination have a necessary and sufficient characterising epistemic state.

Halpern and Moses (1990): Common knowledge is necessary and sufficient for simultaneous actions.

- Necessary: A simultaneous action implies common knowledge of the action.

Review: Knowledge Gain and Coordination

Many forms of coordination have a necessary and sufficient characterising epistemic state.

Halpern and Moses (1990): Common knowledge is necessary and sufficient for simultaneous actions.

- Necessary: A simultaneous action implies common knowledge of the action.
- Sufficient: Common knowledge of any fact arises simultaneously among all agents, and thus can be used to coordinate a simultaneous action.

Review: Knowledge Gain and Coordination

Many forms of coordination have a necessary and sufficient characterising epistemic state.

Ben-Zvi and Moses (2010): Nested knowledge is necessary and sufficient for linear ordering of actions.

Review: Knowledge Gain and Coordination

Many forms of coordination have a necessary and sufficient characterising epistemic state.

Ben-Zvi and Moses (2010): Nested knowledge is necessary and sufficient for linear ordering of actions.

- Necessary:
  $\mathrm{act}_2 \Rightarrow K_2(\text{previously } \mathrm{act}_1)$,
  $\mathrm{act}_3 \Rightarrow K_3 K_2(\text{previously } \mathrm{act}_1)$, etc.

Review: Knowledge Gain and Coordination

Many forms of coordination have a necessary and sufficient characterising epistemic state.

Ben-Zvi and Moses (2010): Nested knowledge is necessary and sufficient for linear ordering of actions.

- Necessary:
  $\text{act}_2 \Rightarrow K_2(\text{previously act}_1)$,
  $\text{act}_3 \Rightarrow K_3 K_2(\text{previously act}_1)$, etc.
- Sufficient: Nested knowledge of any fact arises linearly, and thus can be used to coordinate linearly-ordered actions.

Reminder: The Runs and Systems Model

Based upon Fagin *et al.* (1995)

Reminder: The Runs and Systems Model

Based upon Fagin *et al.* (1995)

- Agents: $\mathbb{I}$.

Reminder: The Runs and Systems Model

Based upon Fagin *et al.* (1995)

- Agents: $\mathbb{I}$.
- Time: $\mathbb{T} = \mathbb{N} \cup \{0\}$. (For ease of presentation.)

Reminder: The Runs and Systems Model

Based upon Fagin *et al.* (1995)

- Agents: $\mathbb{I}$.
- Time: $\mathbb{T} = \mathbb{N} \cup \{0\}$. (For ease of presentation.)
- Runs: $\mathcal{R}$. Each run is a function from time to:
  - A state of the environment — and —
  - A state for each agent.

Reminder: The Runs and Systems Model

Based upon Fagin *et al.* (1995)

- Agents: $\mathbb{I}$.
- Time: $\mathbb{T} = \mathbb{N} \cup \{0\}$. (For ease of presentation.)
- Runs: $\mathcal{R}$. Each run is a function from time to:
  - A state of the environment — and —
  - A state for each agent.
- A protocol governs the agents' behaviour.

Set-Theoretic Reasoning

Set-Theoretic Reasoning

- A point $(r, t) \in \Omega_R \triangleq \mathcal{R} \times \mathbb{T}$ defines a snapshot of the system.

Set-Theoretic Reasoning

- A point $(r, t) \in \Omega_R \triangleq \mathcal{R} \times \mathbb{T}$ defines a snapshot of the system.
- An event is associated with the set of points $\psi \in \mathcal{F}_R \triangleq 2^{\Omega_R}$ at which it holds.

Set-Theoretic Reasoning

- A point $(r, t) \in \Omega_R \triangleq \mathcal{R} \times \mathbb{T}$ defines a snapshot of the system.
- An event is associated with the set of points $\psi \in \mathcal{F}_R \triangleq 2^{\Omega_R}$ at which it holds.
  - $\psi \subseteq \phi$ means "ψ validly implies ϕ".

Set-Theoretic Reasoning

- A point $(r, t) \in \Omega_R \triangleq \mathcal{R} \times \mathbb{T}$ defines a snapshot of the system.
- An event is associated with the set of points $\psi \in \mathcal{F}_R \triangleq 2^{\Omega_R}$ at which it holds.
  - $\psi \subseteq \phi$ means "ψ validly implies ϕ".

For any event ψ,

Set-Theoretic Reasoning

- A point $(r,t) \in \Omega_R \triangleq \mathcal{R} \times \mathbb{T}$ defines a snapshot of the system.
- An event is associated with the set of points $\psi \in \mathcal{F}_R \triangleq 2^{\Omega_R}$ at which it holds.
  - $\psi \subseteq \phi$ means "ψ validly implies ϕ".

For any event ψ,

$$\bigcirc^d \psi \triangleq \{(r,t) \mid (r,t+d) \in \psi\}$$

Set-Theoretic Reasoning

- A point $(r, t) \in \Omega_R \triangleq \mathcal{R} \times \mathbb{T}$ defines a snapshot of the system.
- An event is associated with the set of points $\psi \in \mathcal{F}_R \triangleq 2^{\Omega_R}$ at which it holds.
  - $\psi \subseteq \phi$ means "ψ validly implies ϕ".

For any event ψ,

$$\bigcirc^{d}\psi \triangleq \{(r,t) \mid (r, t+d) \in \psi\}$$

$$\bigcirc^{\leq d}\psi \triangleq \{(r,t) \mid \exists t' \leq t+d : (r, t') \in \psi\}$$

Set-Theoretic Reasoning

- A point $(r,t) \in \Omega_R \triangleq \mathcal{R} \times \mathbb{T}$ defines a snapshot of the system.
- An event is associated with the set of points $\psi \in \mathcal{F}_R \triangleq 2^{\Omega_R}$ at which it holds.
  - $\psi \subseteq \phi$ means "ψ validly implies ϕ".

For any event ψ,

$$\bigcirc^d \psi \triangleq \{(r,t) \mid (r,t+d) \in \psi\}$$

$$\bigcirc^{\leq d} \psi \triangleq \{(r,t) \mid \exists t' \leq t+d : (r,t') \in \psi\}$$

$$K_j \psi \triangleq \{(r,t) \mid [(r,t)]_j \subseteq \psi\}$$

$[(r,t)]_j$ is the event "the local state of j is $r_j(t)$".

Epistemic Fixed-Point Analysis

Epistemic Fixed-Point Analysis

Halpern and Moses (1990)

Epistemic Fixed-Point Analysis

Halpern and Moses (1990)

- Common knowledge of ψ is the greatest fixed point of $x \mapsto E_I(\psi \cap x)$.
    - In the lattice $\mathcal{F}_R$.

Epistemic Fixed-Point Analysis

Halpern and Moses (1990)

- Common knowledge of ψ is the greatest fixed point of $x \mapsto E_I(\psi \cap x)$.
  - In the lattice $\mathcal{F}_R$.
- A few variants:

Epistemic Fixed-Point Analysis

Halpern and Moses (1990)

- Common knowledge of ψ is the greatest fixed point of $x \mapsto E_I(\psi \cap x)$.
  - In the lattice $\mathcal{F}_R$.
- A few variants:
  - Define *Eventual Common Knowledge of* ψ as the greatest fixed point of $x \mapsto \cap_{j \in I} \diamond K_j(\psi \cap x)$.
    - Necessary and sufficient for eventual coordination.

Epistemic Fixed-Point Analysis

Halpern and Moses (1990)

- Common knowledge of ψ is the greatest fixed point of $x \mapsto E_I(\psi \cap x)$.
  - In the lattice $\mathcal{F}_R$.
- A few variants:
  - Define *Eventual Common Knowledge of* ψ as the greatest fixed point of $x \mapsto \cap_{j \in I} \diamond K_j(\psi \cap x)$.
    - Necessary and sufficient for eventual coordination.
  - Define ε*-Common Knowledge of* ψ as the greatest fixed point of $x \mapsto E_I^{\varepsilon}(\psi \cap x)$.
    - Necessary and sufficient for up-to-ε coordination.

Reasoning about Coordination

Reasoning about Coordination

Fagin *et al.* (1995)

An I-ensemble, where $I \subseteq \mathbb{I}$ is a set of agents, is an I-tuple of events $\bar{\mathbf{e}} = (\mathbf{e}_j)_{j \in I} \in {\mathcal{F}_R}^I$, in which for every agent $j \in I$, the event $\mathbf{e}_j$ is a j-local event.

- An event is j-local if it is known to j whenever it occurs.

Reasoning about Coordination

Fagin *et al.* (1995)

An I-ensemble, where $I \subseteq \mathbb{I}$ is a set of agents, is an I-tuple of events $\bar{\mathbf{e}} = (\mathbf{e}_j)_{j \in I} \in {\mathcal{F}_R}^I$, in which for every agent $j \in I$, the event $\mathbf{e}_j$ is a j-local event.

- An event is j-local if it is known to j whenever it occurs.

Central example: the tuple

$$\bar{\mathbf{e}} = \{\text{“}j \text{ is starting to clean right now”}\}_{j \in I}$$

is an ensemble.

Can these be Generalized to deal with Arbitrary Timely Constraints?

Can these be Generalized to deal with Arbitrary Timely Constraints?

Definition (Timely Coordination)

Can these be Generalized to deal with Arbitrary Timely Constraints?

Definition (Timely Coordination)

1. A timely-coordination spec is a pair (I, δ), where $I \subseteq \mathbb{I}$ is a set of agents and $\delta : I^2 \rightarrow \mathbb{Z} \cup \{\infty\}$.

Can these be Generalized to deal with Arbitrary Timely Constraints?

Definition (Timely Coordination)

1. A timely-coordination spec is a pair (I, δ), where $I \subseteq \mathbb{I}$ is a set of agents and $\delta : I^2 \to \mathbb{Z} \cup \{\infty\}$.

$$\begin{aligned} \delta(r,l) &= -7 & \delta(l,r) &= 3 \\ \delta(d,r) &= 6 & \delta(r,d) &= 11 \\ \delta(l,d) &= -9 & \delta(d,l) &= -4 \end{aligned}$$

Can these be Generalized to deal with Arbitrary Timely Constraints?

Definition (Timely Coordination)

1. A timely-coordination spec is a pair (I, δ), where $I \subseteq \mathbb{I}$ is a set of agents and $\delta : I^2 \to \mathbb{Z} \cup \{\infty\}$.
2. Given a timely-coordination spec (I, δ) and a system $R \subseteq \mathcal{R}$, we say that an I-ensemble $\bar{\mathbf{e}} \in \mathcal{F}_R{}^I$ is δ-coordinated (in R) if for every $(j, k) \in I^2$ and for every $(r, t) \in \mathbf{e}_j$, there exists $t' \leq t + \delta(j, k)$ s.t. $(r, t') \in \mathbf{e}_k$.

$$
\begin{aligned}
\delta(r, l) &= -7 & \qquad \delta(l, r) &= 3 \\
\delta(d, r) &= 6 & \qquad \delta(r, d) &= 11 \\
\delta(l, d) &= -9 & \qquad \delta(d, l) &= -4
\end{aligned}
$$

Can these be Generalized to deal with Arbitrary Timely Constraints?

Definition (Timely Coordination)

1. A timely-coordination spec is a pair (I, δ), where $I \subseteq \mathbb{I}$ is a set of agents and $\delta : I^2 \to \mathbb{Z} \cup \{\infty\}$.
2. Given a timely-coordination spec (I, δ) and a system $R \subseteq \mathcal{R}$, we say that an I-ensemble $\bar{\mathbf{e}} \in {\mathcal{F}_R}^I$ is δ-coordinated (in R) if for every $(j, k) \in I^2$ and for every $(r, t) \in \mathbf{e}_j$, there exists $t' \leq t + \delta(j, k)$ s.t. $(r, t') \in \mathbf{e}_k$.

$$\begin{aligned} \delta(r,l) &= -7 & \delta(l,r) &= 3 \\ \delta(d,r) &= 6 & \delta(r,d) &= 11 \\ \delta(l,d) &= -9 & \delta(d,l) &= -4 \end{aligned}$$

Captures Car-Wash, Simultaneous/Ordered Coordination, etc.

Main Analysis Difficulties

- Actions are not necessarily ordered.

Main Analysis Difficulties

- Actions are not necessarily ordered.
  - Dependencies may be circular.

Main Analysis Difficulties

- Actions are not necessarily ordered.
  - Dependencies may be circular.
  - Infinitely many epistemic consequences.

Main Analysis Difficulties

- Actions are not necessarily ordered.
  - Dependencies may be circular.
  - Infinitely many epistemic consequences.
- δ is not necessarily symmetric w.r.t. the agents in I.

Symmetric Forms of Coordination are Easier

In the simultaneous / up-to-ε / eventual coordination scenarios, the constraints are symmetric (invariant under permutations on I):

Symmetric Forms of Coordination are Easier

In the simultaneous / up-to-ε / eventual coordination scenarios, the constraints are symmetric (invariant under permutations on I):

- All agents have to know that within a certain (global) temporal constraint all agents will act.

Symmetric Forms of Coordination are Easier

In the simultaneous / up-to-ε / eventual coordination scenarios, the constraints are symmetric (invariant under permutations on I):

- All agents have to know that within a certain (global) temporal constraint all agents will act.
- All agents have to know that within a certain (global) temporal constraint all agents will have to know that within these temporal constraints all agents will act.

Symmetric Forms of Coordination are Easier

In the simultaneous / up-to-ε / eventual coordination scenarios, the constraints are symmetric (invariant under permutations on I):

- All agents have to know that within a certain (global) temporal constraint all agents will act.
- All agents have to know that within a certain (global) temporal constraint all agents will have to know that within these temporal constraints all agents will act.
- etc.

Symmetric Forms of Coordination are Easier

In the simultaneous / up-to-ε / eventual coordination scenarios, the constraints are symmetric (invariant under permutations on I):

- All agents have to know that within a certain (global) temporal constraint all agents will act.
- All agents have to know that within a certain (global) temporal constraint all agents will have to know that within these temporal constraints all agents will act.
- etc.
- Solutions are greatest fixed point of functions of the form $x \mapsto E^*(\psi \cap x)$, for some "triggering" ψ, and for E^* along the lines of "within a certain temporal constraint, everyone will know that ...".

δ-Coordination is Generally Asymmetric

For general δ, such a fixed point is elusive:

δ-Coordination is Generally Asymmetric

For general δ, such a fixed point is elusive:

- In the car-wash scenario,
  $start_l \Rightarrow K_l$: within -7 to 3: $start_r$.

δ-Coordination is Generally Asymmetric

For general δ, such a fixed point is elusive:

- In the car-wash scenario,
  $start_l \Rightarrow K_l$: within -7 to 3: $start_r$.
- Similarly yet asymmetrically,
  $start_d \Rightarrow K_d$: within -11 to -6: $start_r$.

δ-Coordination is Generally Asymmetric

For general δ, such a fixed point is elusive:

- In the car-wash scenario,
  $start_l \Rightarrow K_l$: within -7 to 3: $start_r$.
- Similarly yet asymmetrically,
  $start_d \Rightarrow K_d$: within -11 to -6: $start_r$.
- What should E^* encompass? -7 to 3 or -11 to -6?

Searching for a Suitable Fixed Point

Let ψ_c be the event "the car c is in the car-wash facility".

$$x \quad \mapsto \quad K_{?}\Big(\psi_c \cap \quad \text{“in ? to ? time units: } x\text{”} \quad \Big)$$

Searching for a Suitable Fixed Point

Let ψ_c be the event "the car c is in the car-wash facility".

$$\text{start}_r \quad \mapsto \quad K_r\left(\psi_c \cap \begin{array}{c}\text{"in } -3 \text{ to } 7 \text{ time units: start}_l\text{"} \cap \\ \text{"in } 6 \text{ to } 11 \text{ time units: start}_d\text{"}\end{array}\right)$$

Searching for a Suitable Fixed Point

Let ψ_c be the event "the car c is in the car-wash facility".

$$\mathsf{start}_l \mapsto K_l\left(\psi_c \cap \begin{array}{c}\text{"in } -7 \text{ to } 3 \text{ time units : } \mathsf{start}_r\text{"} \cap \\ \text{"in } 4 \text{ to } 9 \text{ time units: } \mathsf{start}_d\text{"}\end{array}\right)$$

$$\mathsf{start}_r \mapsto K_r\left(\psi_c \cap \begin{array}{c}\text{"in } -3 \text{ to } 7 \text{ time units: } \mathsf{start}_l\text{"} \cap \\ \text{"in } 6 \text{ to } 11 \text{ time units: } \mathsf{start}_d\text{"}\end{array}\right)$$

$$\mathsf{start}_d \mapsto K_d\left(\psi_c \cap \begin{array}{c}\text{"in } -9 \text{ to } -4 \text{ time units: } \mathsf{start}_l\text{"} \cap \\ \text{"in } -11 \text{ to } -6 \text{ time units: } \mathsf{start}_r\text{"}\end{array}\right)$$

Searching for a Suitable Fixed Point

Let ψ_c be the event "the car c is in the car-wash facility".

$$\mathsf{start}_l \mapsto K_l\left(\psi_c \cap \begin{array}{c}\text{“in } -7 \text{ to } 3 \text{ time units : } \mathsf{start}_r\text{” } \cap \\ \text{“in } 4 \text{ to } 9 \text{ time units: } \mathsf{start}_d\text{”}\end{array}\right)$$

$$\mathsf{start}_r \mapsto K_r\left(\psi_c \cap \begin{array}{c}\text{“in } -3 \text{ to } 7 \text{ time units: } \mathsf{start}_l\text{” } \cap \\ \text{“in } 6 \text{ to } 11 \text{ time units: } \mathsf{start}_d\text{”}\end{array}\right)$$

$$\mathsf{start}_d \mapsto K_d\left(\psi_c \cap \begin{array}{c}\text{“in } -9 \text{ to } -4 \text{ time units: } \mathsf{start}_l\text{” } \cap \\ \text{“in } -11 \text{ to } -6 \text{ time units: } \mathsf{start}_r\text{”}\end{array}\right)$$

A *vectorial* E^*!

Searching for a Suitable Fixed Point

Let ψ_c be the event "the car c is in the car-wash facility".

$$\text{start}_l \mapsto K_l\left(\psi_c \cap \begin{array}{c}\text{"in } -7 \text{ to } 3 \text{ time units : start}_r\text{"} \cap \\ \text{"in } 4 \text{ to } 9 \text{ time units: start}_d\text{"}\end{array}\right)$$

$$\text{start}_r \mapsto K_r\left(\psi_c \cap \begin{array}{c}\text{"in } -3 \text{ to } 7 \text{ time units: start}_l\text{"} \cap \\ \text{"in } 6 \text{ to } 11 \text{ time units: start}_d\text{"}\end{array}\right)$$

$$\text{start}_d \mapsto K_d\left(\psi_c \cap \begin{array}{c}\text{"in } -9 \text{ to } -4 \text{ time units: start}_l\text{"} \cap \\ \text{"in } -11 \text{ to } -6 \text{ time units: start}_r\text{"}\end{array}\right)$$

A *vectorial* E^*!

Searching for a Suitable Fixed Point

Let ψ_c be the event "the car c is in the car-wash facility".

$$\text{start}_l \mapsto K_l\left(\psi_c \cap \begin{array}{c}\text{"in } -7 \text{ to } 3 \text{ time units : start}_r\text{"} \cap \\ \text{"in } 4 \text{ to } 9 \text{ time units: start}_d\text{"}\end{array}\right)$$

$$\text{start}_r \mapsto K_r\left(\psi_c \cap \begin{array}{c}\text{"in } -3 \text{ to } 7 \text{ time units: start}_l\text{"} \cap \\ \text{"in } 6 \text{ to } 11 \text{ time units: start}_d\text{"}\end{array}\right)$$

$$\text{start}_d \mapsto K_d\left(\psi_c \cap \begin{array}{c}\text{"in } -9 \text{ to } -4 \text{ time units: start}_l\text{"} \cap \\ \text{"in } -11 \text{ to } -6 \text{ time units: start}_r\text{"}\end{array}\right)$$

A *vectorial* E^*!

$$(x_j)_{j\in I} \mapsto \left(K_j\left(\psi \cap \bigcap_{k\in I\setminus\{j\}} \bigcirc^{\leq\delta(j,k)} x_k\right)\right)_{j\in I}$$

Timely Common Knowledge

Definition (Timely Common Knowledge)

Timely Common Knowledge

We work in the lattice ${\mathcal{F}_R}^I$.

Definition (Timely Common Knowledge)

Let $R \subseteq \mathcal{R}$ and let (I, δ) be a timely-coordination spec. For each $\psi \in \mathcal{F}_R$, we define δ-common knowledge of ψ by I, denoted by $C_I^\delta \psi$, to be the greatest fixed point of the function

$$\begin{aligned} f_\psi^\delta : \quad {\mathcal{F}_R}^I &\rightarrow {\mathcal{F}_R}^I \\ (x_i)_{i \in I} &\mapsto \left(K_i \left(\psi \cap \bigcap_{j \in I \setminus \{i\}} \bigcirc^{\leq \delta(i,j)} x_j \right) \right)_{i \in I} \end{aligned}$$

Note

$C_I^\delta \psi$ is a tuple.

Lemma

$C_I^\delta \psi$ is well-defined.

Timely Common Knowledge Properties

Timely Common Knowledge Properties

Many Desirable Properties — E.g. Induction Rule

Every $\bar{\xi} \in \mathcal{F}_R{}^I$ *satisfying* $\bar{\xi} \leq f_\psi^\delta(\bar{\xi})$ *also satisfies* $\bar{\xi} \leq C_I^\delta \psi$.

- Recall that $\leq$ means "coordinate-wise validly implies".

Timely Common Knowledge Properties

Many Desirable Properties — E.g. Induction Rule

Every $\bar{\xi} \in {\mathcal{F}_R}^I$ *satisfying* $\bar{\xi} \leq f_\psi^\delta(\bar{\xi})$ *also satisfies* $\bar{\xi} \leq C_I^\delta \psi$*.*

- Recall that $\leq$ means "coordinate-wise validly implies".

Theorem

$C_I^\delta \psi$ *is the largest* δ*-coordinated ensemble s.t.* $(C_I^\delta)_j \subseteq \psi$ *for every agent* $j \in I$*.*

Timely Common Knowledge Properties

Many Desirable Properties — E.g. Induction Rule

Every $\bar{\xi} \in \mathcal{F}_R{}^I$ *satisfying* $\bar{\xi} \leq f_\psi^\delta(\bar{\xi})$ *also satisfies* $\bar{\xi} \leq C_I^\delta \psi$.

- Recall that $\leq$ means "coordinate-wise validly implies".

Theorem

$C_I^\delta \psi$ *is the largest* δ*-coordinated ensemble s.t.* $(C_I^\delta)_j \subseteq \psi$ *for every agent* $j \in I$.

Timely Common Knowledge Properties

Many Desirable Properties — E.g. Induction Rule

Every $\bar{\xi} \in \mathcal{F}_R{}^I$ *satisfying* $\bar{\xi} \leq f_\psi^\delta(\bar{\xi})$ *also satisfies* $\bar{\xi} \leq C_I^\delta \psi$.

- Recall that $\leq$ means "coordinate-wise validly implies".

Theorem

$C_I^\delta \psi$ *is the largest* δ*-coordinated ensemble s.t.* $(C_I^\delta)_j \subseteq \psi$ *for every agent* $j \in I$.

Timely Common Knowledge Properties

Many Desirable Properties — E.g. Induction Rule

Every $\bar{\xi} \in \mathcal{F}_R{}^I$ *satisfying* $\bar{\xi} \leq f_\psi^\delta(\bar{\xi})$ *also satisfies* $\bar{\xi} \leq C_I^\delta \psi$.

- Recall that $\leq$ means "coordinate-wise validly implies".

Theorem

$C_I^\delta \psi$ *is the largest* δ*-coordinated ensemble s.t.* $(C_I^\delta)_j \subseteq \psi$ *for every agent* $j \in I$.

Timely Common Knowledge Properties

Many Desirable Properties — E.g. Induction Rule

Every $\bar{\xi} \in \mathcal{F}_R{}^I$ *satisfying* $\bar{\xi} \leq f_{\psi}^{\delta}(\bar{\xi})$ *also satisfies* $\bar{\xi} \leq C_I^{\delta}\psi$.

- Recall that $\leq$ means "coordinate-wise validly implies".

Theorem

$C_I^{\delta}\psi$ *is the largest* δ*-coordinated ensemble s.t.* $(C_I^{\delta})_j \subseteq \psi$ *for every agent* $j \in I$.

Timely Common Knowledge Properties

Many Desirable Properties — E.g. Induction Rule

Every $\bar{\xi} \in \mathcal{F}_R{}^I$ *satisfying* $\bar{\xi} \le f^{\delta}_{\psi}(\bar{\xi})$ *also satisfies* $\bar{\xi} \le C^{\delta}_I \psi$.

- Recall that $\le$ means "coordinate-wise validly implies".

Theorem

$C^{\delta}_I \psi$ *is the largest* δ*-coordinated ensemble s.t.* $(C^{\delta}_I)_j \subseteq \psi$ *for every agent* $j \in I$.

Observation

The ensemble in which the j*'th event is "*$(C^{\delta}_I)_j$ *holds for the first time" is* δ*-coordinated.*

Applying Timely Common Knowledge

Applying Timely Common Knowledge

(ψ_c is the event "the car c is in the car-wash facility".)

Applying Timely Common Knowledge

(ψ_c is the event "the car c is in the car-wash facility".)

Observation

Every protocol solving the car-wash problem corresponds to a δ-coordinated ensemble $\bar{\mathbf{e}}$ s.t. $\mathbf{e}_j \subseteq \psi_c$ for every agent $j \in I$.

Applying Timely Common Knowledge

(ψ_c is the event "the car c is in the car-wash facility".)

Observation

Every protocol solving the car-wash problem corresponds to a δ-coordinated ensemble $\bar{\mathbf{e}}$ s.t. $\mathbf{e}_j \subseteq \psi_c$ for every agent $j \in I$.

Applying Timely Common Knowledge

(ψ_c is the event "the car c is in the car-wash facility".)

Observation

Every protocol solving the car-wash problem corresponds to a δ-coordinated ensemble $\bar{\mathbf{e}}$ s.t. $\mathbf{e}_j \subseteq \psi_c$ for every agent $j \in I$.

Applying Timely Common Knowledge

(ψ_c is the event "the car c is in the car-wash facility".)

Observation

Every protocol solving the car-wash problem corresponds to a δ-coordinated ensemble $\bar{\mathbf{e}}$ s.t. $\mathbf{e}_j \subseteq \psi_c$ for every agent $j \in I$.

Applying Timely Common Knowledge

(ψ_c is the event "the car c is in the car-wash facility".)

Observation

Every protocol solving the car-wash problem corresponds to a δ-coordinated ensemble $\bar{\mathbf{e}}$ s.t. $\mathbf{e}_j \subseteq \psi_c$ for every agent $j \in I$.

Corollary (Necessary and Sufficient for Solving Car-Wash)

Applying Timely Common Knowledge

(ψ_c is the event "the car c is in the car-wash facility".)

Observation

Every protocol solving the car-wash problem corresponds to a δ-coordinated ensemble $\bar{\mathbf{e}}$ s.t. $\mathbf{e}_j \subseteq \psi_c$ for every agent $j \in I$.

Corollary (Necessary and Sufficient for Solving Car-Wash)

- The protocol in which each robot j starts washing/drying at the first instant at which $(C_I^\delta \psi_c)_j$ holds (during each run) is an optimal solution for the car-wash problem.

Applying Timely Common Knowledge

(ψ_c is the event "the car c is in the car-wash facility".)

Observation

Every protocol solving the car-wash problem corresponds to a δ-coordinated ensemble $\bar{\mathbf{e}}$ s.t. $\mathbf{e}_j \subseteq \psi_c$ for every agent $j \in I$.

Corollary (Necessary and Sufficient for Solving Car-Wash)

- The protocol in which each robot j starts washing/drying at the first instant at which $(C_I^\delta \psi_c)_j$ holds (during each run) is an optimal solution for the car-wash problem.

Applying Timely Common Knowledge

(ψ_c is the event "the car c is in the car-wash facility".)

Observation

Every protocol solving the car-wash problem corresponds to a δ-coordinated ensemble $\bar{\mathbf{e}}$ s.t. $\mathbf{e}_j \subseteq \psi_c$ for every agent $j \in I$.

Corollary (Necessary and Sufficient for Solving Car-Wash)

- The protocol in which each robot j starts washing/drying at the first instant at which $(C_I^\delta \psi_c)_j$ holds (during each run) is an optimal solution for the car-wash problem.

Applying Timely Common Knowledge

(ψ_c is the event "the car c is in the car-wash facility".)

Observation

Every protocol solving the car-wash problem corresponds to a δ-coordinated ensemble $\bar{\mathbf{e}}$ s.t. $\mathbf{e}_j \subseteq \psi_c$ for every agent $j \in I$.

Corollary (Necessary and Sufficient for Solving Car-Wash)

- The protocol in which each robot j starts washing/drying at the first instant at which $(C_I^\delta \psi_c)_j$ holds (during each run) is an optimal solution for the car-wash problem.
- The car-wash problem is solvable in a run r iff there exists t s.t. $(r,t) \in (C_I^\delta \psi_c)_j$ for some/all j.

Applying Timely Common Knowledge

(ψ_c is the event "the car c is in the car-wash facility".)

Observation

Every protocol solving the car-wash problem corresponds to a δ-coordinated ensemble $\bar{\mathbf{e}}$ s.t. $\mathbf{e}_j \subseteq \psi_c$ for every agent $j \in I$.

Corollary (Necessary and Sufficient for Solving Car-Wash)

- The protocol in which each robot j starts washing/drying at the first instant at which $(C_I^\delta \psi_c)_j$ holds (during each run) is an optimal solution for the car-wash problem.
- The car-wash problem is solvable in a run r iff there exists t s.t. $(r, t) \in (C_I^\delta \psi_c)_j$ for some/all j.

We prove this for a general class of Timely-Coordinated Response (TCR) problems that we define. (General I, δ, actions, etc.)

The Iterative Definition of Common Knowledge

Definition (Common Knowledge - Popular Definiton)

Let $R \subseteq \mathcal{R}$ and let $I \subseteq \mathbb{I}$. For every $\psi \in \mathcal{F}_R$,

$$C_I\psi \triangleq \bigcap_{n=1}^{\infty} {E_I}^n\psi,$$

where ${E_I}^0\psi \triangleq \psi$ and ${E_I}^n\psi \triangleq E_I {E_I}^{n-1}\psi$ for every $n \in \mathbb{N}$.

The Iterative Definition of Common Knowledge

Definition (Common Knowledge - Popular Definiton)

Let $R \subseteq \mathcal{R}$ and let $I \subseteq \mathbb{I}$. For every $\psi \in \mathcal{F}_R$,

$$C_I\psi \triangleq \bigcap_{n=1}^{\infty} {E_I}^n\psi,$$

where ${E_I}^0\psi \triangleq \psi$ and ${E_I}^n\psi \triangleq E_I{E_I}^{n-1}\psi$ for every $n \in \mathbb{N}$.

Equivalently,

$$C_I\psi = \bigcap_{(i_1,\ldots,i_n)\in I^*} K_{i_1}\cdots K_{i_n}\psi$$

where I^* is the set of all finite sequences of elements of I.

A Generalization for Timely-Common Knowledge

Theorem (Nested-Knowledge Characterisation)

A Generalization for Timely-Common Knowledge

Theorem (Nested-Knowledge Characterisation)

Under natural assumptions (mainly perfect recall), we have:

$$(C_I^\delta \psi)_i \;=\; \bigcap_{(i,i_2,\ldots,i_n)\in I^*} K_i \bigcirc^{\delta(i,i_2)} K_{i_2} \bigcirc^{\delta(i_2,i_3)} K_{i_3} \cdots \bigcirc^{\delta(i_{n-1},i_n)} K_{i_n}\psi,$$

A Generalization for Timely-Common Knowledge

Theorem (Nested-Knowledge Characterisation)

Under natural assumptions (mainly perfect recall), we have:

$$(C_I^\delta \psi)_i \;=\; \bigcap_{(i,i_2,\ldots,i_n)\in I^*} K_i \bigcirc^{\delta(i,i_2)} K_{i_2} \bigcirc^{\delta(i_2,i_3)} K_{i_3} \cdots \bigcirc^{\delta(i_{n-1},i_n)} K_{i_n}\psi,$$

- In the general case,
  the r.h.s. above is *not* sufficient for solving TCR, while the fixed-point definition always is.

A Generalization for Timely-Common Knowledge

Theorem (Nested-Knowledge Characterisation)

Under natural assumptions (mainly perfect recall), we have:

$$(C_I^\delta \psi)_i \;=\; \bigcap_{(i,i_2,\ldots,i_n)\in I^*} K_i \bigcirc^{\delta(i,i_2)} K_{i_2} \bigcirc^{\delta(i_2,i_3)} K_{i_3} \cdots \bigcirc^{\delta(i_{n-1},i_n)} K_{i_n}\psi,$$

- In the general case,
  the r.h.s. above is *not* sufficient for solving TCR, while the fixed-point definition always is.
- An iterative definition that does not require any assumptions is uglier. (Involves a definition by transfinite induction.)

A Generalization for Timely-Common Knowledge

Theorem (Nested-Knowledge Characterisation)

Under natural assumptions (mainly perfect recall), we have:

$$(C_I^\delta \psi)_i \;=\; \bigcap_{(i,i_2,\ldots,i_n)\in I^*} K_i \bigcirc^{\delta(i,i_2)} K_{i_2} \bigcirc^{\delta(i_2,i_3)} K_{i_3} \cdots \bigcirc^{\delta(i_{n-1},i_n)} K_{i_n}\psi,$$

- In the general case, the r.h.s. above is *not* sufficient for solving TCR, while the fixed-point definition always is.
- An iterative definition that does not require any assumptions is uglier. (Involves a definition by transfinite induction.)
- In many naturally-occurring models, timely common knowledge is attainable (and thus TCR is solvable) even when common knowledge is not. (E.g. the two generals.)

Summary

- The connection between coordination and epistemology extends deeper than realized so far.

Summary

- The connection between coordination and epistemology extends deeper than realized so far.
- Loose circular codependence between actions introduces significant complexity.

Summary

- The connection between coordination and epistemology extends deeper than realized so far.
- Loose circular codependence between actions introduces significant complexity.
- Asymmetric coordination necessitates a leap to vectorial fixed points.

Summary

- The connection between coordination and epistemology extends deeper than realized so far.
- Loose circular codependence between actions introduces significant complexity.
- Asymmetric coordination necessitates a leap to vectorial fixed points.
- Timely Common Knowledge strictly generalizes common knowledge and previous variants.

Summary

- The connection between coordination and epistemology extends deeper than realized so far.
- Loose circular codependence between actions introduces significant complexity.
- Asymmetric coordination necessitates a leap to vectorial fixed points.
- Timely Common Knowledge strictly generalizes common knowledge and previous variants.
- Vectorial fixed points are a natural, powerful tool for protocol analysis.

Questions?

Thank you!